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On Radon maps with values in arbitrary topological vector spaces, and their integral extensions

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Introduction.

When K is a compact topological space and E is a Banach space a continuous linear map $\mu : \mathcal{C}(K) \longrightarrow E$ has a 'natural' extension to the bounded Borel functions if and only if it is weakly compact. There are other conditions equivalent to this (such as: μ maps weakly compact sets into compact sets) which continue to make sense, and to be equivalent, even when E is a complete (not necessarily locally convex) topological vector space, and when these conditions hold an integration process, whereby μ is extended to a class $\mathcal{L}^1(\mu)$ containing the bounded Borel functions, is possible which is entirely analogous to the one used by N. Bourbaki [3] to establish Riesz' representation theorem and related results. In the locally convex case weak compactness criteria for the map μ are, by duality, equivalent to weak compactness criteria for families of scalar measures equipped with an appropriate topology. A. Grothendieck has given several weak compactness criteria (see [6]) applicable to maps from spaces $\mathcal{C}(K)$ or families of measures, and we have been able to establish the analogues of these in the case of arbitrary quasi complete topological vector spaces. When E is weakly sequentially complete an arbitrary map $\mu : \mathcal{C}(K) \longrightarrow E$ is weakly compact. A. Pelczynski [8] and C. Bessaga and Pelczynski [2] have shown that an arbitrary map is from a space $\mathcal{C}(K)$ to a Banach space E is weakly compact if and only if all sequences (x_n) in E for which the finite sums $\sum_n \lambda_n x_n$ are bounded when $|\lambda_n| \leq 1$, are summable in E (weak sequential completeness implies this by Orlicz theorem). We have shown below that when E is an arbitrary quasi-complete topological vector space the same condition is still equivalent to the extendibility of every μ, for example to the fact that an arbitrary μ maps weakly compact subsets into compact subsets. This implies, thanks to work of L. Schwartz [11] that an arbitrary map from $\mathscr{C}(K)$ to a space L^p, $0 \leq p < +\infty$ possesses the extension property. For $1 \leq p < +\infty$ this was already known because these spaces are weakly sequentially complete (even reflexive for $p > 1$ so that weak compactness is trivial in those cases). The space $L^0(\nu)$ is the space of ν-measurable functions with the topology of convergence in measure on sets of finite measure. Conceivably some of these results could have applications to the theory of stochastic integrals.

Let us point out here a difference between locally convex spaces and arbitrary topological vector spaces which has a considerable effect on this article: In a complete locally convex space it follows from the summability of a family $(x_i)_{i \in I}$ that for any bounded family of scalars $(\lambda_i x_i)_{i \in I}$ is also summable. We owe to S. Rolewicz and C. Ryll-Nardzewski [10] the observation that this is no longer true for even an arbitrary complete metrisable space E. At the same time these authors show that for a locally bounded, or more generally a locally pseudo-convex space, the equivalence still holds. This accounts for the fact that in the present article integrals $\int g f d\mu$ are often considered, where g is a bounded Borel function, rather than as usual integrals $\int_A f d\mu$. In many cases where it is known that E is locally pseudo-convex it would be sufficient to consider 'indefinite' integrals of the type $\int_A f d\mu$, To facilitate the exposition we have termed B-summable those families $(x_i)_{i \in I}$ for which $(\lambda_i x_i)_{i \in I}$ is summable for all bounded scalar families $(\lambda_i)_{i \in I} \in \ell^\infty(I)$.

Rather than maps μ defined on a space $\mathcal{C}(K)$ we consider maps $\mu : \mathcal{K}(T) \longrightarrow E$ where T is a locally compact Hausdorff space, and $\mathcal{K}(T)$ is the space of continuous functions with compact support. The map μ is termed a Radon map when for every compact set $K \subset T$ the restriction of μ to the space $\mathcal{K}(T, K)$ of functions with support in K is continuous relatively to the topology of uniform convergence.

In §1 we define the space $\mathcal{L}^1(\mu)$ of μ-integrable functions where μ is a Radon map. This paragraph contains statements one can make about arbitrary Radon maps.

The term Radon measure is reserved for those Radon maps for which $\mathcal{L}^1(\mu)$ contains the bounded Borel functions with compact support. These are introduced in §2 and basic convergence theorems (2.8-12) are established for them. An integrability criterion (2-13) linking integrability with discrete summability is proved here.

§3 gives a number of conditions equivalent to weak compactness in the locally convex case. It is shown that μ is a Radon measure if $\mathcal{L}^1(\mu)$ contains all functions χ_K where K is a compact G_δ, but an example is given of a Radon map $\mu : \mathcal{C}[0,1] \to E$ with values in a Banach space, for which $\mathcal{L}^1(\mu)$ contains all step functions, hence the ruled functions, but which is not a Radon measure.

In §1, 2, 3 only Radon maps with values in metrisable topological vector spaces have been considered. This restriction is lifted in §4. Essentially no new problems occur since every topological vector space is a subspace of a product of metrisable spaces. Theorem 4-3 recapitulates some of the properties obtained thus far in a manner which is valid for arbitrary quasi-complete t.v. spaces. An entirely different construction of $\mathcal{L}^1(\mu)$, independent of any choice of metric in E is given at the end of §4. In §5 the quasi-complete spaces E with the property that every E-valued Radon map is a Radon measure are characterised. In §6 'weak' integration is used to obtain practical integrability criteria. It is assumed here that the quasi-complete t.v.s. E is plunged into a t.v-s F, with continuous linear injection $E \underset{j}{\hookrightarrow} F$, and μ-integrability is compared to $\tilde{\mu}$- integrability, where $\tilde{\mu} = j \circ \mu$.

§1 Definition of $\mathcal{L}^1(\mu)$ and $\mathcal{L}^0(\mu)$.

In what follows we assume that E is a complete metrisable topological vector space (t.v.s.) over $\mathbb{C}$. Accordingly E has a basis of neighborhoods of zero of the form $\{x : |x| \leq \varepsilon\}$, $\varepsilon > 0$, where the 'invariant metric' $x \longrightarrow |x|$ has the properties: $|x+y| \leq |x| + |y|$, $|\lambda x| \leq |x|$ for $|\lambda| \leq 1$, $\lambda \in \mathbb{C}$, $\lim_{\substack{\lambda \to 0 \\ \lambda \in \mathbb{C}}} |\lambda x| = 0$, and $|x| = 0$ implies $x = 0$. (Without the last property this function would be called an invariant pseudo-metric). Conversely given such an invariant metric on a linear space E, it defines a topology which is compatible with the vector space structure, i.e. for which the maps $(x, y) \longrightarrow x + y$ and $(\lambda, x) \longrightarrow \lambda x$ are continuous. We assume that on E such a metric is chosen once for all.

Let T be a locally compact Hausdorff space. We denote by φ, ψ continuous complex functions with compact support, and the set of these is denoted by $\mathcal{K}(T)$. The letter A stands for a subset of T, K for a compact subset, χ_A for the characteristic function of A.

A Radon map $\mu : \mathcal{K}(T) \longrightarrow E$ is a linear map with the following continuity property: For every $\varepsilon > 0$ and compact K there is $\delta > 0$ such that $\sup_{t \in T} |\varphi(t)| \leq \delta$ and support $\varphi \subset K$ implies $|\mu(\varphi)| \leq \varepsilon$.

1.1 Definition Given a function $f : T \longrightarrow [0, +\infty]$ we put:

$$\dot{\mu}(f) = \sup_{|\varphi| \leq f} |\mu(\varphi)| \quad \text{when } f \text{ is lower semi-continuous (l.s.c.)}$$

$$\mu^{\bullet}(f) = \inf_{\substack{f \leq g \\ g \text{ l.s.c.}}} \mu^{\bullet}(g) \qquad \text{when } f \text{ has compact support,}$$

and for arbitrary f :

$$\mu^{\bullet}(f) = \sup_{h \leq f} \mu^{\bullet}(h) \qquad \text{where } h \text{ has compact support.}$$

The function $f \longrightarrow \mu^{\bullet}(f)$ will be called the semi-variation of μ (1) .

When A is a subset of T we put $\mu^{\bullet}(A) = \mu^{\bullet}(\chi_A)$; a set for which $\mu^{\bullet}(A) = 0$ will be termed μ-negligible and the expression μ-almost everywhere (μ-a.e.) will be used as in measure theory. The coherence of the above definition will be proved in connection with the following theorem:

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(1) We use the notation $\mu^{\bullet}$ of N. Bourbaki [3] and the term semi-variation of R. G. Bartle, N. Dunford and J. Schwartz [1]. The relation between the semi-variation of those authors and the above $\mu^{\bullet}$ is discussed in [14] p. 132.

1.2 <u>Theorem</u> a) $f \leq g$ implies $\dot{\mu}(f) \leq \dot{\mu}(g)$; $\dot{\mu}(0) = 0$. If $(f_i)_{i \in I}$ is an increasingly directed family of lower semi-continuous functions with upper bound f,

$$\dot{\mu}(f) = \sup_{i \in I} \dot{\mu}(f_i)$$

b) $\dot{\mu}(f_1 + f_2) \leq \dot{\mu}(f_1) + \dot{\mu}(f_2)$ and more generally

$$\dot{\mu}(\sum_n f_n) \leq \sum_n \dot{\mu}(f_n) \quad \text{and} \quad \dot{\mu}(\bigcup_n A_n) \leq \sum_n \dot{\mu}(A_n)$$

for any sequence of functions f_n or sets A_n.

c) $\dot{\mu}(f) = 0$ is equivalent to $f(t) = 0$ μ a.e. and $f(t) = g(t)$ μ.a.e implies $\dot{\mu}(f) = \dot{\mu}(g)$.

d) If f is a bounded function with compact support, we have $\dot{\mu}(f) < +\infty$ and $\lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \dot{\mu}(\lambda f) = 0$.

1.3 <u>Lemma</u> For any lower semi-continuous function f $\dot{\mu}(f) = \sup_{0 \leq \varphi \ll f} \dot{\mu}(\varphi)$ where $\varphi \ll f$ means $\varphi \leq f$, support $\varphi \subset \{t : f(t) > 0\}$ and $\varphi(t) < f(t)$ whenever $\varphi(t) > 0$.

<u>Proof</u>. Let $\lambda < \dot{\mu}(f)$. Then there exists $\psi \in \mathcal{K}(T)$ such that $|\psi| \leq f$ $\lambda < |\mu(\psi)| \leq \dot{\mu}(f)$. Let $\omega : \{t : f(t) > 0\}$ and choose ε such that $0 < \varepsilon < |\mu(\psi)| - \lambda$. Assume $|\psi_1| \leq |\psi|$ and $|\psi - \psi_1| < \delta$ implies $|\mu(\psi_1) - \mu(\psi)| \leq \varepsilon$. Since ψ tends to zero at the boundary of ω it is possible to find ψ_1 satisfying the above conditions and such that $\operatorname{supp} \psi_1 \subset \omega$, and for $\alpha < 1$ sufficiently close to 1 the function $\alpha \psi_1$ satisfies the same

conditions. Then $\varphi = |\alpha\psi_1| << f$ and $\lambda < |\mu(\alpha\psi_1)| \leq \dot{\mu}(\varphi) \leq \dot{\mu}(f)$, which proves the lemma.

This lemma implies in particular that the above definition is coherent.

We now prove the theorem, the first assertion of which is obvious. Let $f_i \uparrow f$, let $\lambda < \dot{\mu}(f)$ and chose $\varphi << f$ such that $\lambda \leq \dot{\mu}(\varphi)$. Then if $K = \operatorname{supp}\varphi$ and $\omega_i = \{t : f_i(t) > \varphi(t)\}$, $\cup\omega_i = \omega \supset K$, hence $\omega_i \supset K$ for some i, and consequently $f_i \geq \varphi$. Then $\lambda \leq \dot{\mu}(\varphi) \leq \dot{\mu}(f_i) \leq \dot{\mu}(f)$ proving a). Now to prove $\dot{\mu}(f_1 + f_2) \leq \dot{\mu}(f_1) + \dot{\mu}(f_2)$ when f_1 and f_2 are l.s.c it is sufficient by a) to consider the case where $f_1, f_2 \in \mathcal{K}(T)$. Then if $\varphi \in \mathcal{K}(T)$ with $|\varphi| \leq f_1 + f_2$ there exist $\varphi_1, \varphi_2 \in \mathcal{K}(T)$ such that $\varphi = \varphi_1 + \varphi_2$ and $|\varphi_i| \leq f_i$. Thus $|\mu(\varphi)| \leq |\mu(\varphi_1)| + |\mu(\varphi_2)| \leq \dot{\mu}(f_1) + \dot{\mu}(f_2)$ and φ being arbitrary $\dot{\mu}(f_1 + f_2) \leq \dot{\mu}(f_1) + \dot{\mu}(f_2)$. More generally finite subadditivity results and it now follows from a) that $\dot{\mu}(\sum_n f_n) \leq \sum_n \dot{\mu}(f_n)$ when (f_n) is a sequence of lower semi-continuous functions. For the general case let K be a compact set and let g_n be lower semi-continuous with $\chi_K f_n \leq g_n$ and $\dot{\mu}(g_n) \leq \dot{\mu}(f_n) + \varepsilon/2^n$. Then if $g = \sum g_n$ and $f = \sum f_n$, $\chi_K f \leq g$ and

$$\dot{\mu}(\chi_K f) \leq \dot{\mu}(g) \leq \sum_n \dot{\mu}(g_n) \leq \sum \dot{\mu}(\chi_K f_n) + \varepsilon \leq \sum \dot{\mu}(f_n) + \varepsilon$$

and since $\dot{\mu}(f) = \sup_K \dot{\mu}(\chi_K f)$, and $\varepsilon > 0$ is arbitrary the inequality $\dot{\mu}(f) \leq \sum \dot{\mu}(f_n)$ is proved. A fortiori $\dot{\mu}(\cup A_n) \leq \dot{\mu}(\sum_n \chi_{A_n}) \leq \sum_n \dot{\mu}(A_n)$ whence b).

c) Let $A = \{t : f(t) > 0\}$ and $A_n = \{t : f(t) \geq \frac{1}{n}\}$, n integer and assume $\dot{\mu}(f) = 0$. We have $\chi_{A_n} \leq nf$ whence

$\mu^\bullet(A_n) \leq \mu^\bullet(nf) \leq n\,\mu^\bullet(f) = 0$ (finite subadditivity). Hence $\mu^\bullet(A) = \mu^\bullet(\cup A_n) = 0$. Conversely assume $\mu^\bullet(A) = 0$ and let $f_n = \inf(f,n)$. Then $f_n \leq n\chi_A$ whence $\mu^\bullet(f_n) \leq \mu^\bullet(n\chi_A) \leq n\mu^\bullet(A) = 0$ and finally $\mu^\bullet(f) \leq \mu^\bullet(\sum_n f_n) \leq \sum_n \mu^\bullet(f_n) = 0$ whence $\mu^\bullet(f) = 0$. If $h = \inf(f,g)$ $h(t) = f(t)$ a.e $\mu^\bullet(h) \leq \mu^\bullet(f) \leq \mu^\bullet(h) + \mu^\bullet(f-h) = \mu^\bullet(h)$; similarly $\mu^\bullet(h) = \mu^\bullet(g)$.

d) Let ω be an open set with compact closure and let $\varepsilon > 0$ be given. There exists an integer $n > 0$ such that $|\varphi| \leq \frac{1}{n}\chi_\omega$ implies $|\mu(\varphi)| \leq \varepsilon$ whence $\mu^\bullet(\frac{1}{n}\chi_\omega) \leq \varepsilon$. Now if $0 \leq f \leq m\chi_\omega$ for some integer m, $\mu^\bullet(f) \leq \mu^\bullet(mn\frac{1}{n}\chi_\omega) \leq mn\varepsilon < +\infty$, while $0 < \lambda \leq \frac{1}{mn}$ implies $\mu^\bullet(\lambda f) \leq \mu^\bullet(\frac{1}{n}\chi_\omega) \leq \varepsilon$ whence c).

1.3 <u>Definition</u> Let $\mathcal{L}^1(\mu)$ be the set of functions $f : T \longrightarrow \mathbb{C}$ such that for every $\varepsilon > 0$ there exists $\varphi \in \mathcal{K}(T)$ for which $\mu^\bullet(|f-\varphi|) \leq \varepsilon$.

1.4 <u>Theorem</u> a) $f, g \in \mathcal{L}^1(\mu)$ implies $f + g \in \mathcal{L}^1(\mu)$ and $hf \in \mathcal{L}^1(\mu)$ for h bounded continuous, in particular $\mathcal{L}^1(\mu)$ is a linear space. Furthermore $f \in \mathcal{L}^1(\mu)$ implies Ref, Imf, $|f| \in \mathcal{L}^1(\mu)$ and if f is real valued $f^+, f^- \in \mathcal{L}^1(\mu)$.

b) $\mu^\bullet(|f+g|) \leq \mu^\bullet(|f|) + \mu^\bullet(|g|)$ and $\lim\limits_{\lambda \to 0} \mu^\bullet(|\lambda f|) = 0$ for $f, g \in \mathcal{L}^1(\mu)$. $\mathcal{L}^1(\mu)$ is a topological vector space with a neighborhood basis of zero composed of the sets $\{f : \mu^\bullet(|f|) \leq \varepsilon\}$ $\varepsilon > 0$.

c) The set $N(\mu)$ of complex functions f such that $\mu^\bullet(|f|) = 0$, or equivalently $f(t) = 0$ μ.a.e, is a closed subspace of $\mathcal{L}^1(\mu)$

and the quotient space $L^1(\mu) = \mathcal{L}^1(\mu)/_{N(\mu)}$ is a complete metric topological vector space.

d) The map μ has a unique continuous extension $f \longrightarrow \int f d\mu$ to $\mathcal{L}^1(\mu)$, with values in E and

$$\dot{\mu}(|f|) = \sup_{|\varphi| \leq 1} \left| \int \varphi f d\mu \right| \qquad \forall f \in \mathcal{L}^1(\mu),$$

in particular:

$$\left| \int f d\mu \right| \leq \dot{\mu}(|f|)$$

<u>Proof</u> $\dot{\mu}(|f+g - (\varphi + \Psi)|) \leq \dot{\mu}(|f-\varphi|) + \dot{\mu}(|g-\Psi|)$.
If $|h| \leq 1$ $\dot{\mu}(|hf - h\varphi|) \leq \dot{\mu}(|f-\varphi|)$. Thus $f+g$, $hg \in \mathcal{L}^1(\mu)$. When $|h| \leq n$, (n integer) $|\frac{h}{n}| \leq 1$ and $hf = n\frac{h}{n}f \in \mathcal{L}^1(\mu)$. The inequalities $|\mathrm{Re} f - \mathrm{Re}\,\varphi| \leq |f-\varphi|$, $||f| - |\varphi|| \leq |f-\varphi|$, $|f^+ - \varphi^+| \leq |f-\varphi|$ etc. prove the last part of a). The first inequality in b) follows from Theorem 1.2 a) and b). For the second relation let $\varepsilon > 0$ be given and choose $\varphi \in \mathcal{K}$ with $\dot{\mu}(|f-\varphi|) \leq \varepsilon/2$. Then for $|\lambda| \leq \delta \leq 1$, $\dot{\mu}(|\lambda f|) \leq \dot{\mu}(|f-\varphi|) + \dot{\mu}(|\lambda\varphi|) \leq \varepsilon$ for δ sufficiently small, by Theorem 1.2 d) thus the assertion in b) follows.

c) $N(\mu)$ is clearly a closed subspace, whence $L^1(\mu)$ is separated and metrisable. We need only prove that $\mathcal{L}^1(\mu)$ is complete and for this it suffices to prove that every 'normally convergent' series is convergent: Let $\sum_n \dot{\mu}(|f_n|) < +\infty$ and put $f(t) = \sum_n f_n(t)$ when $\sum |f_n(t)| < +\infty$ and $f(t) = 0$ elsewhere. Let $s_n = \sum_{i=1} f_i$. Then $|f(t) - s_n(t)| \leq \sum_{i>n} |f_i(t)|$ for all t, hence $\dot{\mu}(|f-s_n|) \leq \sum_{i>n} \dot{\mu}(|f_i|)$ by 1.2 b, whence $\lim_n \dot{\mu}(|f-s_n|) = 0$ as was to be proved.

d) Since $|\mu(\varphi)| \leq \mu^{\bullet}(|\varphi|)$ μ is continuous for the topology induced on $\mathcal{K}(T)$ by $\mathcal{L}^1(\mu)$, consequently $\mathcal{K}(T)$ being a dense subspace by construction, a unique linear extension, denoted by $f \longrightarrow \int f d\mu$, exists, and the above inequality extends to $|\int f d\mu| \leq \mu^{\bullet}(|f|)$, since the metric $x \longrightarrow |x|$ and the pseudo metric $f \longrightarrow \mu^{\bullet}(|f|)$ are continuous on E respectively on $\mathcal{L}^1(\mu)$ $(||x| - |y|| \leq |x-y|)$. Thus if we put $N(f) = \sup_{|\varphi| \leq 1} |\int \varphi f d\mu|$ we have $N(f) \leq \mu^{\bullet}(|f|)$ and since $|\int \varphi(f_1+f_2) d\mu| \leq |\int \varphi f_1 d\mu| + |\int \varphi f_2 d\mu| \leq N(f_1) + N(f_2)$ for $|\varphi| \leq 1$ it follows that $N(f_1 + f_2) \leq N(f_1) + N(f_2)$ and consequently $|N(f_1) - N(f_2)| \leq N(f_1 - f_2) \leq \mu^{\bullet}(|f_1 - f_2|)$, whence N is continuous on $\mathcal{L}^1(\mu)$. It is sufficient therefore to prove $N(f) = \mu^{\bullet}(|f|)$ for $f \in \mathcal{K}(T)$. But this results immediately from the definition and the following fact: every $\psi \in \mathcal{K}(T)$ with $|\psi| \leq |f|$ can be approximated uniformly by functions of the form φf where $|\varphi| \leq 1$, $\varphi \in \mathcal{K}(T)$ and $\operatorname{supp} \varphi$ is contained in some compact neighborhood of $\operatorname{supp} f$. This proves the theorem.

<u>Remark</u> The topological vector space $\mathcal{L}^1(\mu)$ does not depend on the choice of metric on E. Indeed for two such metrics $|\ |_1$ and $|\ |_2$ and the corresponding semi-variations $\mu_1^{\bullet}$ and $\mu_2^{\bullet}$ it is easily checked that if $|x|_1 < \delta$ implies $|x|_2 \leq \varepsilon$, $\mu_1^{\bullet}(f) < \delta$ implies $\mu_2^{\bullet}(f) \leq \varepsilon$.

Observe further that if f is some complex function such that for every $\varepsilon > 0$ there exists $g \in \mathcal{L}^1(\mu)$ with $\mu^{\bullet}(|f-g|) \leq \varepsilon$, f also belongs to $\mathcal{L}^1(\mu)$, (by virtue of the triangle inequality).

In particular if $\mu^\bullet(|f-g|) = 0$, i.e. if $f(t) = g(t)$ μ.a.e, f belongs to $\mathcal{L}^1(\mu)$, and since $|\int f d\mu - \int g d\mu| \leq \mu^\bullet(|f-g|) = 0$, $\int f d\mu = \int g d\mu$. This leads to the introduction of the usual definition:

1.5 <u>Definition</u> Any function f with values in $\mathbb{C}$ or $\overline{\mathbb{R}}$ will be called μ-integrable if it coincides μ.a.e with a function g belonging to $\mathcal{L}^1(\mu)$. We put $\int f d\mu = \int g d\mu$ by definition.

Observe also that if $(f_n)_{n \geq 1}$ is a sequence such that $\lim_n \mu^\bullet(|f_n|) = 0$, there exists a subsequence $(f_{n_k})_{k \geq 1}$ such that $\lim_k f_{n_k}(t) = 0$ μ.a.e. Indeed if $\sum_n \mu^\bullet(|f_n|) < +\infty$ it follows that $\lim_n f_n(t) = 0$ μ.a.e for $\mu^\bullet(\overline{\lim}|f_n|) \leq \mu^\bullet(\sum_{p \geq n} |f_p|) \leq \sum_{p \geq n} \mu^\bullet(|f_p|)$ for all n, hence $\mu^\bullet(\overline{\lim}|f_n|) = 0$ and consequently $\overline{\lim}|f_n(t)| = 0$ μ.a.e.

For the next theorem we need the following results of some independent interest:

1.6 <u>Proposition</u> Let $f : T \longrightarrow [0, +\infty]$ be μ-integrable

a) For every $\varepsilon > 0$ there exists K compact such that $\mu^\bullet(\chi_{\complement K} f) \leq \varepsilon$.

b) For every $\varepsilon > 0$ there exists δ such that $\mu^\bullet(A) \leq \delta$ implies $\mu^\bullet(\chi_A f) \leq \varepsilon$.

<u>Proof.</u> There is $\varphi \in \mathcal{K}(T)$ such that $\mu^\bullet(|f-\varphi|) \leq \varepsilon/2$, whence $\mu^\bullet(\chi_A f) \leq \mu^\bullet(|f-\varphi|) + \mu^\bullet(\chi_A|\varphi|) \leq \varepsilon/2 + \mu^\bullet(\chi_A|\varphi|)$. Then for $A = \complement K$ with $K = \operatorname{supp}\varphi$ $\mu^\bullet(\chi_{\complement K} f) \leq \varepsilon/2 < \varepsilon$, and if $|\varphi| \leq n$ $\mu^\bullet(\chi_A f) \leq \varepsilon/2 + n\,\mu^\bullet(A) \leq \varepsilon$ provided $\mu^\bullet(A) \leq \delta = \varepsilon/2n$.

1.7 <u>Proposition</u> a) When ω is open $\mu^\bullet(\omega) = \sup_{K \subset \omega} \mu^\bullet(K)$, K compact.

b) When A has a compact closure $\mu^\bullet(A) = \inf_{A \subset \omega} \mu^\bullet(\omega)$ where ω is open.

c) For arbitrary A, $\mu^\bullet(A) = \sup_K \mu^\bullet(A \cap K)$.

<u>Proof</u> Let $\lambda < \mu^\bullet(\omega)$. Then by lemma 1.3 there exists $\varphi \leq \chi_\omega$ with supp $\varphi = K \subset \omega$ and $\lambda < \mu^\bullet(\varphi)$, since $0 \leq \varphi \leq \chi_K$ $\lambda < \mu^\bullet(K)$ proving a). Let A be as given, then by definition of $\mu^\bullet(A)$ there exists f lower semi-continuous such that $\chi_A \leq f$ and $\mu^\bullet(f) < \mu^\bullet(A) + \varepsilon$. If ω is a relatively compact neighborhood of A we may (upon replacement of f by $\inf(f, \chi_\omega)$) assume $f \leq \chi_\omega$. Then by Theorem 1.2 b and d $\mu^\bullet(\lambda f)$ depends continuously on λ, whence $\mu^\bullet(\lambda f) \leq \mu^\bullet(A) + \varepsilon$ for some $\lambda > 1$. Let $\omega = \{t : \lambda f(t) > 1\}$. Then ω is open, $\omega \supset A$, and $\chi_\omega \leq \lambda f$, whence $\mu^\bullet(\omega) \leq \mu^\bullet(A) + \varepsilon$, and ε being arbitrary this proves b). Property c) results directly from the definition since $\mu^\bullet(f) = \sup_K \mu(\chi_K f)$ for all $f \geq 0$ (if $h \leq f$ has compact support K, $h \leq \chi_K f \leq f$); this ends the proof.

In particular one may at this stage define the support of μ to be the complement of the largest μ-negligible open set.

1.8 <u>Definition</u> A function $f : T \longrightarrow \mathbb{C}$ (or $\overline{\mathbb{R}}$ or some topological space) is said to possess the <u>Lusin property</u> when for every compact K and $\varepsilon > 0$ there exists $K' \subset K$ with $\mu^\bullet(K-K') \leq \varepsilon$ such that the restriction $f/_{K'}$ of f to K' is continuous. We denote by $\mathcal{L}^0(\mu)$ the set of complex functions possessing the Lusin property.

Clearly if f belongs to $\mathcal{L}^0(\mu)$ so does $|f|$, Re f, Im f, f^+.

If f and g belong to $\mathcal{L}^{o}(\mu)$ we can find $K' \subset K$ with $\mu^{*}(K - K') \leq \varepsilon/2$ then $K'' \subset K'$ with $\mu^{*}(K' - K'') \leq \varepsilon/2$, such that $f/_{K'}$ and $g/_{K''}$ are continuous. A fortiori: $(f+g)/_{K''}$ and $fg/_{K''}$ are continuous whence $f + g$ and $fg \in \mathcal{L}^{o}(\mu)$. Every continuous function belongs to $\mathcal{L}^{o}(\mu)$. Furthermore if f possesses the Lusin property and $g(t) = f(t)$ μ.a.e so does g. For let $A = \{t : f(t) \neq g(t)\}$, let K be a given compact, $K' \subset K$ with $\mu^{*}(K - K') \leq \varepsilon/2$ such that $f/_{K'}$ is continuous and let ω be an open set containing $A \cap K'$ such that $\mu^{*}(\omega) \leq \varepsilon/2$ then if $K'' = K' \cap \complement\omega$ $\mu^{*}(K - K'') \leq \varepsilon$ and $g/_{K''}$ is continuous.

1.9 <u>Theorem</u> Every μ - integrable function possesses the Lusin property, in particular $\mathcal{L}^{1}(\mu) \subset \mathcal{L}^{o}(\mu)$. Conversely if $f \in \mathcal{L}^{o}(\mu)$ and $|f(t)| \leq g(t)$ μ-a.e. where $g \in \mathcal{L}^{1}(\mu)$ it follows that $f \in \mathcal{L}^{1}(\mu)$. In particular if $g \in \mathcal{L}^{o}(\mu)$ is bounded and $f \in \mathcal{L}^{1}(\mu)$, $gf \in \mathcal{L}^{1}(\mu)$.

<u>Proof</u> Let $f \in \mathcal{L}^{1}(\mu)$ and let $\psi_n \in \mathcal{K}(T)$ with $\mu^{*}(|f - \psi_n|) \leq 1/_{2^{n+2}}$ for $n \geq 1$. Then $\psi_n(t)$ tends to $f(t)$ μ.a.e and we may assume $f(t) = \lim\limits_{n} \psi_n(t)$ wherever the limit exists. Put $\varphi_n = \psi_n - \psi_{n-1}$, $\varphi_o = 0$. Then $f(t) = \sum\limits_{n \geq 1} \varphi_n(t)$, wherever the series converges and $\mu^{*}(|\varphi_n|) \leq \frac{1}{2^n}$. Let $h(t) = \sum\limits_{n=1}^{\infty} n|\varphi_n(t)| \leq +\infty$, and let $O_a = \{t : h(t) > a\}$, so that O_a is open since h is lower semi-continuous. Assume $a \geq 1$. Then since $\chi_{O_a} \leq \frac{1}{a} h \leq \sum\limits_{n \geq 1} \frac{n}{a}|\varphi_n|$ we have

$$\mu^\bullet(O_a) \le \sum_{n=1}^{\infty} n\, \mu^\bullet(\tfrac{1}{a}|\varphi_n|) \le \sum_{n=1}^{N} n\, \mu^\bullet(\tfrac{1}{a}|\varphi_n|) + \sum_{n=N+1}^{+\infty} \frac{n}{2^n} \le \varepsilon$$

provided N is first chosen so that the second part is smaller than $\varepsilon/2$, and a is subsequently chosen large enough for the first part to be smaller than $\varepsilon/2$. Thus $\lim_{a\to+\infty} \mu^\bullet(O_a) = 0$. Now for $t \in \complement O_a$, $n \sum_{k\ge n} |\varphi_k(t)| \le \sum_{k\ge n} k|\varphi_k(t)| \le h(t) \le a$ so that $\Sigma|\varphi_k(t)|$ and a fortiori $\sum_k \varphi_k(t) = f(t)$ converges uniformly on $\complement O_a$, whence the restriction of f to $\complement O_a$ is continuous, and a fortiori $f \in \mathcal{L}^0(\mu)$. For the proof of the converse we may assume $0 \le f \le g$, and we first consider the particular case where f is bounded and has compact support, whence $0 \le f \le \varphi$ for some $\varphi \in \mathcal{K}(T)$. Let $K = \operatorname{supp}\varphi$ and let $K' \subset K$ with $\mu^\bullet(K-K') \le \frac{\varepsilon}{2n}$. (where n is an integer such that $\varphi \le n$), and such that $f/_{K'}$ is continuous. Let $\bar{f}$ be a nonnegative continuous extension of $f/_{K'}$ to T (apply Urysohn's theorem in the one point compactification of T) and let $\psi = \inf(\varphi, \bar{f})$. Then $\psi \in \mathcal{K}(T)$, $\psi(t) = f(t)$ for $t \in K'$, and $0 \le \psi \le \varphi$, whence $\mu^\bullet(|f-\psi|) \le \mu^\bullet(\chi_{\complement K}|f-\psi|) + \mu^\bullet(\chi_{K'}|f-\psi|) + \mu^\bullet(\chi_{K-K'}|f-\psi|) \le \mu^\bullet(\chi_{K-K'}|f-\psi|) \le \mu^\bullet(\chi_{K-K'}\, 2n)$ $\le 2n\, \mu^\bullet(K-K') \le \varepsilon$, $|f-\psi|$ being zero on $\complement K$ and K', which proves that $f \in \mathcal{L}^1(\mu)$. Next consider the general case where $0 \le f \le g$. Let $f_n = \inf(f,n)$ and $A_n = \{t : f(t) \ge n\}$. Then $n\chi_{A_n} \le f \le g$ whence $\mu^\bullet(A_n) \le \mu^\bullet(\frac{1}{n} g)$ and consequently $\lim_{n\to\infty} \mu^\bullet(A_n) = 0$. Now $\mu^\bullet(f-f_n) = \mu^\bullet(\chi_{A_n}(f-f_n)) \le \mu^\bullet(\chi_{A_n} g)$ so that $\lim_{n\to\infty} \mu^\bullet(f-f_n) = 0$ by proposition 1.6. Thus it suffices to prove that f_n belongs to $\mathcal{L}^1(\mu)$ and we may and shall assume that the given function f is bounded. Let K be a compact such that $\mu^\bullet(\chi_{\complement K}\, 2g) \le \varepsilon$ (prop. 1.6) and let $\varphi \in \mathcal{K}_+(T)$ be chosen such that $f(t) \le \varphi(t)$ for $t \in K$.

Then $h = \inf(\varphi, f)$ belongs to $\mathcal{L}^1(\mu)$ according to the previously considered case, and $h(t) = f(t)$ for $t \in K$, whence $\mu^\bullet(|f-h|) = \mu^\bullet(\chi_{\complement K}|f-h|) \leq \mu^\bullet(\chi_{\complement K}(2g) \leq \varepsilon$, proving that $f \in \mathcal{L}^1(\mu)$ as asserted. Next we prove two convergence theorems:

1.10 Theorem Let $(f_i)_{i \in I}$ be an increasingly directed family of lower semi-continuous functions with upper bound f, and values in $[0, +\infty]$. Then $\mu^\bullet(f) = \sup_{i \in I} \mu(f_i)$ and if f is μ-integrable $\inf_{i \in I} \mu^\bullet(f-f_i) = 0$. In particular if furthermore $f_i \in \mathcal{L}^1(\mu)$, $\int f_i d\mu$ tends to $\int f d\mu$.

Proof The first assertion was already proved in connection with Theorem 1.2. Let $f \geq 0$ be μ-integrable. Choose K compact such that $\mu^\bullet(\chi_{\complement K} f) < \varepsilon$ and such that $f/_K$ is finite continuous, which is possible by prop. 1.6. Then if $0 < \alpha < 1$
$\mu^\bullet(f - \alpha\chi_K f) \leq \mu^\bullet(f - \chi_K f) + \mu^\bullet((1-\alpha)\chi_K f) \leq \varepsilon$ for α sufficiently near 1, by Theorem 1.2 d. Let $A_i = \{t \in K : \alpha f(t) < f_i(t)\}$. Then A_i is open in K, and $A_i \uparrow K$. Thus there exists i such that $A_i = K$ and consequently $\alpha\chi_K f \leq f_i$, whence $\mu^\bullet(f - f_i) \leq \varepsilon$, which proves the second part. Finally if $f_i \in \mathcal{L}^1(\mu)$,

$$\left|\int f d\mu - \int f_i d\mu\right| \leq \mu^\bullet(f - f_i) \leq \varepsilon \quad \text{for}$$

f_i sufficiently large. This ends the proof.

1.11 Theorem Let $(f_i)_{i \in I}$ be a net composed of μ-integrable functions, and let f be some function. Then $\lim_i \mu^\bullet(|f_i - f|) = 0$ if and only if the following three conditions are fulfilled:

i) For every compact K and $\varepsilon > 0$,

$$\lim_i \mu^{\bullet}\{t \in K : |f_i(t) - f(t)| \geq \varepsilon\} = 0$$

ii) For every $\varepsilon > 0$ there is some compact K such that $\mu^{\bullet}(\chi_{\complement K}|f_i|) \leq \varepsilon$ for all i.

iii) For every $\varepsilon > 0$ and compact K there is $\delta > 0$ such that $A \subset K$ and $\mu^{\bullet}(A) \leq \delta$ implies $\mu^{\bullet}(\chi_A|f_i|) \leq \varepsilon$ for all i.

Proof The conditions are necessary: let $A_i : \{t : |f_i(t) - f(t)| \geq \varepsilon\}$ and n be an integer such that $\frac{1}{n} \leq \varepsilon$. Then $\frac{1}{n}\chi_{A_i} \leq |f_i - f|$, and consequently $\mu^{\bullet}(A_i) \leq \mu^{\bullet}(n|f_i - f|) \leq n\,\mu^{\bullet}(|f_i - f|)$ whence $\lim_i \mu^{\bullet}(A_i) = 0$, a fortiori $\lim_i \mu^{\bullet}(A_i \cap K) = 0$. Conditions ii) and iii) follow from proposition 1.6 and the inequality

$$\mu^{\bullet}(\chi_A|f_i|) \leq \mu^{\bullet}(|f_i - f|) + \mu^{\bullet}(\chi_A|f|).$$

Naturally we describe condition i) by saying that f_i tends to f 'in measure' on every compact set. To prove the converse it suffices, as in the traditional proof for the L^P spaces (1), to prove that f_i is a Cauchy net, since then by Theorem 1.4 c f_i converges in $\mathcal{L}^1(\mu)$ to some g, and by the first part of this

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(1) The case of L^p space is indeed a particular case of this since $\mathcal{L}^p(\nu) = \mathcal{L}^1(\mu)$ where $\mu : \mathfrak{X} \longrightarrow L^p$ is the natural inclusion.

proof it follows that f_i tends to g 'in measure' on every compact K, whence it follows easily that $g(t) = f(t)$ μ-a.e and consequently that f_i tends to f in $\mathcal{L}^1(\mu)$. Now given $\varepsilon > 0$ we prove that $\dot{\mu}(|f_i - f_j|) \leq 4\varepsilon$ for i, j sufficiently large. First choose K compact such that $\dot{\mu}(\chi_{\complement K}|f_i|) \leq \varepsilon/2$, and let $A_{ij} = \{t \in K : |f_i(t) - f_j(t)| \geq \varepsilon'\}$ where $\dot{\mu}(\varepsilon'\chi_K) \leq \varepsilon$.

Then

$$\dot{\mu}(|f_i - f_j|) \leq \dot{\mu}(\chi_{\complement K}|f_i - f_j|) + \mu(\chi_{K-A_{ij}}|f_i - f_j|) + \dot{\mu}(\chi_{A_{ij}}|f_i - f_j|)$$

$$\leq \varepsilon + \varepsilon + \dot{\mu}(\chi_{A_{ij}}|f_i - f_j|) \leq 4\varepsilon$$

provided $\dot{\mu}(A_{ij}) \leq \delta$ (associated with ε by hypothesis iii), which is the case for i and j sufficiently large since $\dot{\mu}(A_{ij}) \leq \dot{\mu}\{t \in K : |f_i - f| \geq \varepsilon/2\} + \dot{\mu}\{t \in K : |f_j - f| \geq \varepsilon/2\}$ which tends to zero as i and j increase indefinitely.

The preceding properties are of course quite general and nothing for instance guarantees that $\mathcal{L}^1(\mu)$ or $\mathcal{L}^o(\mu)$ contains anything but continuous functions. If $\mu : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$ is the identity map $\mathcal{L}^1(\mu) = \mathcal{L}^o(\mu) = \mathcal{C}(K)$, theorem 1.10 is the classical Dini lemma, and 1.11 is empty. If T is locally compact non compact and $\mu : \mathcal{K}(T) \hookrightarrow \mathcal{C}_o(T)$ is the natural injection into the space of continuous functions tending to zero at infinity $\mathcal{L}^1(\mu) = \mathcal{C}_o(T)$, $\mathcal{L}^o(\mu)$ is the space of all continuous functions, convergence 'in measure on every compact' is uniform convergence on compact sets.

<u>Remark</u>

If for $\varphi \in \mathcal{K}_+(T)$ we put $P(\varphi) = \dot{\mu}(\varphi)$ the functional

$p : \mathcal{K}_+ \longrightarrow \mathbb{R}_+$ satisfies the following properties

a) $\varphi \leq \psi$ implies $p(\varphi) \leq p(\psi)$

b) $p(\varphi + \psi) \leq p(\varphi) + p(\psi)$

c) $\lim\limits_{\lambda \to 0} p(\lambda \varphi) = 0.$

By Theorem 1.2 p determines $\mu^{\bullet}$ completely. Conversely let p be a function with values in $[0, +\infty)$ satisfying a) b) c). Then there exists a complete metric t.v.s. E and a Radon map $\mu : \mathcal{K}(T) \longrightarrow E$ such that $p(\varphi) = \mu^{\bullet}(\varphi)$ for all $\varphi \geq 0$. Indeed if we put $p(\varphi) = p(|\varphi|)$ for $\varphi \in \mathcal{K}(T)$ $\varphi \longrightarrow p(\varphi)$ is an invariant pseudo-metric on $\mathcal{K}(T)$ and if E is the completion of the associated Hausdorff space it is easily checked that the canonical map $\mu : \mathcal{K}(T) \longrightarrow E$ is a Radon map, and that $\mu^{\bullet}(\varphi) = p(\varphi)$ for all $\varphi \in K_+$.

In the next paragraph we confine our attention to Radon maps for which $\mathcal{L}^1(\mu)$ contains at least the bounded Borel functions with compact support, and later we characterise the spaces E such that every E-valued Radon map has this property.

§2 Radon measures

We continue to consider the case where E is a complete metric t.v.s.

2.1 Theorem The following conditions are equivalent:

a) $\mathcal{L}^1(\mu)$ contains all bounded Borel functions with compact support.

b) $\chi_K \in \mathcal{L}^1(\mu)$ for every compact K.

c) Every complex Borel function belongs to $\mathcal{L}^o(\mu)$, i.e possesses the Lusin property.

2.2. Definition A Radon measure is a map μ possessing the above properties.

Proof c) implies a) by theorem 1.9 and a) obviously implies b). To see that a) implies c) it suffices to consider a real valued Borel function f with compact support and to prove that it belongs to $\mathcal{L}^o(\mu)$ (for if $\varphi f \in \mathcal{L}^o(\mu)$ for all $\varphi \in \mathcal{K}(T)$, obviously $f \in \mathcal{L}^o(\mu)$. Let $h : \overline{\mathbb{R}} \longrightarrow [0,1]$ be a homeomorphism. Then $h \circ f$ is a Borel function which belongs to $\mathcal{L}^1(\mu)$ by hypothesis a), and consequently belongs to $\mathcal{L}^o(\mu)$ (Theorem 1.9). Thus $f = h^{-1} \circ h \circ f$ also possesses the Lusin property. It thus only remains to prove that b) implies c). This will be a consequence of the next theorem in the proof of which we will assume only hypothesis b):

2.3 <u>Theorem</u> Let μ be a Radon measure. Let $\mathcal{B}^{\mu}$ be the set of all subsets A of T such that $\chi_{A\cap K} \in \mathcal{L}^1(\mu)$ for all compact K. Then $\mathcal{B}^{\mu}$ is a σ-algebra containing the Borel sets and the following conditions on $f : T \longrightarrow \mathbb{C}$ (or $\overline{\mathbb{R}}$) are equivalent:

i) $f^{-1}(B) \in \mathcal{B}^{\mu}$ for every Borel set $B \subset \mathbb{C}$ (resp $\overline{\mathbb{R}}$).

ii) For every compact K and every $\varepsilon > 0$ there exists $K' \subset K$ with $\mu^{\bullet}(K - K') \leq \varepsilon$ such that $f/_{K'}$ is continuous (i.e. f possesses the Lusin property).

iii) For every compact K there exists a partition $K = N + \sum_n K_n$ where N is μ-negligible and where K_n is compact such that $f/_{K_n}$ is continuous.

2.4 <u>Definition</u> When μ is a Radon measure the sets $A \in \mathcal{B}^{\mu}$ will be called <u>μ-measurable sets</u> and the numerical functions satisfying i) ii) or iii) will be called <u>μ-measurable functions</u>.

<u>Proof</u>. By hypothesis b) in Theorem 2.1, $\mathcal{B}^{\mu}$ contains the closed sets, thus all Borel sets if we prove that it is a σ-algebra, and once we prove that i) implies ii) it will follow that b) implies c). Now since $\chi_{A\cap B\cap K} = \inf(\chi_{A\cap K}, \chi_{B\cap K}) = \dfrac{\chi_{A\cap K} + \chi_{B\cap K} - |\chi_{A\cap K} - \chi_{B\cap K}|}{2}$ and $\chi_{K\cap \complement A} = \chi_K - \chi_{A\cap K}$, $\mathcal{B}^{\mu}$ is an algebra. To prove that it is a σ-algebra we need the following lemmas.

<u>Lemma 1</u> Let f be a numerical function which possesses the Lusin property and let F be a closed subset of $\mathbb{C}$ (resp $\overline{\mathbb{R}}$). Then $f^{-1}(F) \in \mathcal{B}^{\mu}$.

<u>Proof</u> Let K be compact, $\varepsilon > 0$, $K' \subset K$ a compact such that $f/_{K'}$ is continuous and $\mu^{\bullet}(K - K') \leq \varepsilon$. Let $A = f^{-1}(F)$. Then $A \cap K' = \{t \in K' : f(t) \in F\}$ is closed in K' hence compact and $A \cap K - A \cap K' \subset K - K'$ whence $\mu^{\bullet}(\chi_{A \cap K} - \chi_{A \cap K'}) \leq \varepsilon$, which proves that $\chi_{A \cap K} \in \mathcal{L}^1(\mu)$ and consequently $A \in \mathcal{B}^{\mu}$.

<u>Lemma 2</u> Let A_n be a sequence of subsets belonging to $\mathcal{B}^{\mu}$ such that $A_n \subset K$ for some compact K. Let $f(t) = \sum_{n \geq 1} \frac{1}{2^n} \chi_{A_n}(t)$. Then $f \in \mathcal{L}^1(\mu)$.

<u>Proof</u> Let $f_n = \sum_{i=1}^{n} \frac{1}{2^i} \chi_{A_i}$, then $f_n \in \mathcal{L}^1(\mu)$ and $|f - f_n| \leq \frac{1}{2^n} \chi_K$ so that $\mu^{\bullet}(|f - f_n|) \leq \mu^{\bullet}(\frac{1}{2^n} \chi_K)$ whence $\lim_n \mu^{\bullet}(|f - f_n|) = 0$, and $f \in \mathcal{L}^1(\mu)$.

We now prove that $\mathcal{B}^{\mu}$ is a σ - algebra. Let $A_n \in \mathcal{B}^{\mu}$ and put $A = \bigcup_{n \geq 1} A_n$. It suffices to prove that $A \cap K = \bigcup_n (A_n \cap K)$ belongs to $\mathcal{B}^{\mu}$, thus we may assume $A_n \subset K$ for all n. Let $f(t) = \sum_{n \geq 1} \frac{1}{2^n} \chi_{A_n}(t)$. Then since $f \in \mathcal{L}^1(\mu)$ f possesses the Lusin property and by Lemma 1 $\complement A = \{ t : f(t) \leq 0\}$ belongs to $\mathcal{B}^{\mu}$, whence $A \in \mathcal{B}^{\mu}$.

Lemma 1 now shows that every function which possesses the Lusin property is $\mathcal{B}^{\mu}$ - measurable i.e. satisfies condition i). Conversely assume f is $\mathcal{B}^{\mu}$ - measurable (which is the case if f is a Borel function) and let us show that f possesses the Lusin property. Without restricting the generality we may assume that f is real valued, and by composing with a homeomorphism $h : \overline{\mathbb{R}} \to [0,1]$ we may assume $0 \leq f \leq 1$. Finally since it suffices to show that $\varphi f \in \mathcal{L}^0(\mu)$ for all $\varphi \in \mathcal{K}(T)$ and since φf is also

$\mathcal{B}^{\mu}$ - measurable (φ being a Borel function), we may assume that f has compact support, whence $0 \leq f \leq \chi_K$. Then there exists a sequence (f_n) of $\mathcal{B}^{\mu}$ - measurable simple functions with $0 \leq f_n \leq \chi_K$ such that $|f - f_n| \leq \frac{1}{n}$, and since $f_n \in \mathcal{L}^1(\mu)$ and $\mu^{\bullet}(|f - f_n|) \leq \mu^{\bullet}(\frac{1}{n}\chi_K)$ tends to zero it follows that $f \in \mathcal{L}^1(\mu)$ and a fortiori $f \in \mathcal{L}^{\mathrm{o}}(\mu)$. This proves the equivalence between i) and ii) and also terminates the proof of theorem 2.1. It remains to show that iii) is equivalent to the other conditions. However this will follow more easily after the next proposition and lemma will have been established.

2.5 Proposition Let A be a μ-integrable set, i.e. a set for which $\chi_A \in \mathcal{L}^1(\mu)$. Then for every $\varepsilon > 0$ there exists a compact $K \subset A$ such that $\mu^{\bullet}(A-K) \leq \varepsilon$.

Proof By 1.6 there exists a compact set H with $\mu^{\bullet}(A \cap \complement H) \leq \varepsilon/2$. Let $H' \subset H$ be chosen such that $\chi_{A/H'}$ is continuous and $\mu^{\bullet}(H-H') \leq \varepsilon/2$. Then $K = A \cap H' = \{t \in H' : \chi_A(t) \geq 1\}$ is compact and $A \cap H - A \cap H' \subset H - H'$ whence $\mu^{\bullet}(A-K) \leq \varepsilon$.

2.6 Lemma Let $A_n, A \in \mathcal{B}^{\mu}$.

a) If $A_n \supset A_{n+1} \dots, \chi_{A_1} \in \mathcal{L}^1(\mu)$ and $A = \bigcap_n A_n$ is μ-negligible, $\lim_{n \to \infty} \mu^{\bullet}(A_n) = 0$.

b) If $A_n \subset A_{n+1} \dots \subset A$ with $\chi_A \in \mathcal{L}^1(\mu)$, and $A - \bigcup_n A_n$ is μ-negligible $\lim_n \mu^{\bullet}(A-A_n) = 0$.

Proof b) is obviously a consequence of a) which we prove. Replacing A_n by $A_n - A$ we may assume $\cap A_n = \emptyset$. Let $K_n \subset A_n$

with $\mu^{\bullet}(A_n - K_n) \leq \varepsilon/_{2^n}$, and let $H_n = \bigcap_{i=1}^{n} K_i$. Then $A_n - H_n = \bigcap_{i=1}^{n} A_i - \bigcap_{i=1}^{n} K_i \subset \bigcup_{i=1}^{n} (A_i - K_i)$ whence $\mu^{\bullet}(A_n - H_n) \leq \varepsilon$. But $\bigcap_n H_n = \emptyset$, thus there exists n such that $H_n = \emptyset$ and consequently $\mu^{\bullet}(A_n) \leq \varepsilon$, which ends the proof.

Let us now prove the equivalence of conditions ii) and iii) in Theorem 2.3. Clearly iii) implies ii) by the above lemma and the fact that if $H_n = \sum_{i=1}^{n} K_i$ $f/_{H_n}$ is continuous. If ii) is assumed and a μ-integrable set A is given we can, using proposition 2.5, inductively define a sequence of disjoint compact subsets (K_n) contained in A such that $f/_{K_n}$ is continuous and $\mu^{\bullet}(A-H_n) \leq \frac{1}{n}$ where $H_n = \sum_{i=1}^{n} K_i$ (to construct K_{n+1} take $K' \subset A - H_n$ with $\mu^{\bullet}(A - H_n - K') \leq \frac{1}{2(n+1)}$ and then $K_{n+1} \subset K'$ such that $f/_{K_{n+1}}$ is continuous and $\mu^{\bullet}(K' - K_{n+1}) < \frac{1}{2(n+1)}$ whence $\mu^{\bullet}(A - H_{n+1}) \leq \frac{1}{n+1}$. Then if $N = A - \sum_n K_n$ $\mu^{\bullet}(N) \leq \frac{1}{n}$ for all n and $\mu^{\bullet}(N) = 0$, proving a somewhat stronger condition:

2.7) iv) For any μ-integrable set A there exists a partition $A = N + \sum_n K_n$ such that N is μ-negligible and K_n is a compact with $f/_{K_n}$ continuous.

In particular if $1 \in \mathcal{L}^1(\mu)$ we have such a decomposition for the whole space T.

2.8 Theorem Let μ be a Radon measure. Let f_n be a sequence of μ-measurable functions converging μ.a.e. to f. Let A be a μ-integrable set (i.e. $\chi_A \in \mathcal{L}^1(\mu)$). Then for every $\varepsilon > 0$ there exists a compact $K \subset A$ with $\mu^\bullet(A - K) \leq \varepsilon$ such that $f_n(t) \longrightarrow f(t)$ uniformly on K and such that the restriction of each f_n to K is continuous, in particular f is μ-measurable (1).

Proof Using the Lusin property and proposition 2.5 we can inductively define a sequence $(K_n)_{n \geq 0}$ with $K_{n+1} \subset K_n \subset K_0 \subset A$ such that $f_n/_{K_n}$ is continuous and such that $\mu^\bullet(A - K_0) \leq \varepsilon/_4$ and $\mu^\bullet(K_n - K_{n+1}) \leq \frac{1}{2^n}\, \varepsilon/_4$.

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(1)
This statement is valid for f_n and f with values in an arbitrary metric space provided μ-measurable is taken to be synonymous with the Lusin property.

Let $K' = \cap K_n$. Then $\mu^*(A - K') \leq \varepsilon/2$ and $f_n/_{K'}$ is continuous. Let $K_{n,p} = \{t \in K' : \text{dist}(f_q(t), f_r(t)) \leq \frac{1}{p}, \forall\ q, r \geq n\}$. Then $K_{n,p}$ is compact, $K_{n,p} \subset K_{n+1,p} \subset K'$ and since f_n converges a.e on K', $\bigcup_n K_{n,p}$ is almost all of K'. Thus by Lemma 2.6 $\lim_{n\to\infty} \mu^*(K' - K_{n,p}) = 0$ and we can choose an increasing sequence $(n_p)_{p\geq 1}$ such that $\mu^*(K' - K_{n_p,p}) \leq \varepsilon/2^{p+1}$. Let $K'' = \bigcap_p K_{n_p,p}$; then $K' - K'' \subset \bigcup_{p\geq 1} K' - K_{n_p,p}$ whence $\mu^*(K' - K'') \leq \varepsilon/2$ and $\mu^*(A - K'') \leq \varepsilon$. But for $t \in K''$ $\text{dist}(f_q(t), f_r(t)) \leq \frac{1}{p}$ for $q, r \geq n_p$, so that f_n converges uniformly to f on K''. In particular $f/_{K''}$ is continuous. Then if μ is a Radon measure we may take for A any compact set consequently f is μ-measurable.

2.9 <u>Corollary</u> Under the same hypothesis f_n tends to f 'in measure' on every compact set, i.e.

$$\lim_{n\to\infty} \mu^*\{t \in K : |f_n(t) - f(t)| \geq \varepsilon\} = 0 .$$

<u>Proof</u> Let $A_n = \{t \in K : |f_n(t) - f(t)| \geq \varepsilon\}$ and let $\delta > 0$ be given. Choose $K' \subset K$ with $\mu^*(K - K') \leq \delta$ such that f_n converges to f uniformly on K', whence $|f_n(t) - f(t)| < \varepsilon$ for $n \geq N(\varepsilon)$. Then $A_n \subset K - K'$, and $\mu^*(A_n) \leq \delta$ for $n \geq N(\varepsilon)$.

2.10 <u>Theorem</u> Let μ be a Radon measure. Let $(f_n)_{n\geq 1}$ be a sequence of μ-integrable functions converging μ almost everywhere to f. Then f is μ-integrable and $\lim_{n\to\infty} \mu^*(|f - f_n|) = 0$ if and only if the following two conditions are satisfied:

a) For every $\varepsilon > 0$ there exists K compact such that

$\mu^*(\chi_{\complement K}|f_n|) \leq \varepsilon$ for all n.

b) For every $\varepsilon > 0$ and every compact K there is $\delta > 0$ such that $A \subset K$ $\mu^\bullet(A) \leq \delta$ implies $\mu^\bullet(\chi_A |f_n|) \leq \varepsilon$ for all n.

This follows immediately from Theorem 1.11 and the above corollary.

2.11 Corollary (Dominated convergence theorem). Let $(f_n)_{n \geq 1}$ be a sequence of μ-integrable functions converging to f μ.a.e and assume there exists $g \in \mathcal{L}^1(\mu)$ such that $|f_n(t)| \leq g(t)$ μ.a.e. Then f is μ-integrable and $\lim_{n \to \infty} \mu^\bullet(|f_n - f|) = 0$. In particular

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu$$

Hereafter we shall say that a family $(x_i)_{i \in I}$ is B-summable when for any bounded family of scalars $(\lambda_i x_i)_{i \in I}$ is summable. For sequences $(x_n)_{n \in N}$ this is equivalent to the fact that the series $\sum_{n=1}^{\infty} \lambda_n x_n$ converges for every $(\lambda_n) \in \ell^\infty$.(1)

2.12 Corollary Let f be μ-integrable. Then for any sequence $(A_n)_{n \geq 1}$ of disjoint μ-measurable sets with union A, the sequence $(\int_{A_n} fd\mu)_{n \geq 1}$ is B-summable and $\int_A fd\mu = \sum_n \int_{A_n} fd\mu$. (Here $\int_A fd\mu$ stands as usual for $\int \chi_A fd\mu$).

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(1) Recall that in a quasi-complete t.v.s E a family $(x_i)_{i \in I}$ is summable with sum S if for every neighborhood of zero V there exists a finite set K_o such that $K \supset K_o$ implies $S_K - S \in V$, where $S_K = \sum_{i \in K} x_i$. This is equivalent to the Cauchy condition: for every V there is K_o such that $K \cap K_o = \emptyset$ implies $S_K \in V$. In particular any subfamily $(x_i)_{i \in J}$ of a summable family is summable, and $(x_i)_{i \in I}$ is summable iff every countable subfamily is summable. A summable sequence is not necessarily B-summable, unless E is locally pseudo convex. See S. Rolewicz and C. Ryll-Nardzewski, Coll. Math. XVII '67.

2.13 Theorem Let $\mu : \mathcal{K}(T)) \longrightarrow E$ be a Radon measure. Then a function $f : T \longrightarrow \mathbb{C}$ is μ-integrable if and only if a) f is μ-measurable and b) For every sequence $(K_n)_{n \geq 1}$ of disjoint compact sets such that $f/_{K_n}$ is continuous, the sequence $(\int_{K_n} f d\mu)_{n \geq 1}$ is B-summable [1].

Proof a) and b) are obviously necessary conditions. For the converse observe first that a) and b) imply that for any sequence $(K_n)_{n \geq 1}$ of pairwise disjoint

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[1] If E is locally pseudo-convex, in particular if E is locally convex or if for some p with $0 < p \leq 1$ $|\lambda x| = |\lambda|^p |x|$ for all $x \in E$ $\lambda \in \mathbb{C}$, B summable is equivalent to summable. In that case b) can be replaced by the weaker condition b'): For any such sequence (K_n) $\lim_n \int_{K_n} f d\mu = 0$. For applying b') to a sequence $H_n = \sum_{i \in P_n} K_i$, with P_n pairwise disjoint subsets of $\mathbf{N}$, it is seen that the Cauchy condition is satisfied.

compact sets such that $\chi_{K_n} f \in \mathcal{L}^1(\mu)$, the sequence $(\int_{K_n} fd\mu)_n$ is B-summable. For K_n admits a partition $K_n = N_n + \sum_i K_n^i$ such that N_n is μ-negligible and $f/_{K_n^i}$ is continuous. Then the family $(\int_{K_n^i} f\, d\mu)_{n,i}$ is B-summable and a fortiori the family $(\int_{K_n} fd\mu)_n$ since $\int_{K_n} fd\mu = \sum_i \int_{K_n^i} fd\mu$. Next we prove the following lemma.

Lemma For any sequence of disjoint compact sets $(K_n)_n$ such that $\chi_{K_n} f \in \mathcal{L}^1(\mu)$ $\lim_n \dot{\mu}(\chi_{K_n}|f|) = 0$.

Proof Assume on the contrary that $\dot{\mu}(\chi_{K_n}|f|) > \alpha > 0$ for infinitely many, or for simplicity, all n. Then by 1.4d there exist Borel functions g_n such that $|g_n| \leq \chi_{K_n}$ and $|\int g_n fd\mu| > \alpha$, and by approximating g_n by simple functions and using the regularity properties of μ, we can attain that $g_n = \sum_i \alpha_n^i \chi_{K_n^i}$ where $(K_n^i)_i$ is a finite family of disjoint compact subsets of K_n, and $|\alpha_n^i| \leq 1$. Thus $|\sum_i \alpha_n^i \int_{K_n^i} fd\mu| > \alpha$. But by the previous observation the family $(\int_{K_n^i} fd\mu)_{i,n}$ is B-summable which contradicts the above inequality. This proves the lemma. We now proceed with the proof of the theorem; proving first that $\chi_K f \in \mathcal{L}^1(\mu)$ for all compact sets K. Let $K = N + \sum K_i$ where N is μ-negligible and $f/_{K_n}$ is continuous. Let P_n be a sequence of finite disjoint subsets of $\mathbb{N}$ and let $H_n = \sum_{i \in P_n} K_i$. Then by the lemma $\lim_n \dot{\mu}(\chi_{H_n}|f|) = \lim_n \dot{\mu}(\sum_{i \in P_n} \chi_{K_i}|f|) = 0$. Thus by the Cauchy condition the secuence

$(\chi_{K_i}|f|)_i$ is summable in $\mathcal{L}^1(\mu)$ and since its sum is $\chi_K|f|$ a.e, it follows that $\chi_K|f| \in \mathcal{L}^1(\mu)$, hence that $\chi_K f \in \mathcal{L}^1(\mu)$ since f is assumed to be μ-measurable. Finally we prove that $\inf_K \mu^\bullet(|f - \chi_K f|) = 0$, where K is compact. Assume on the contrary $\mu^\bullet(|f - \chi_K f|) > \alpha > 0$ for all compact K. Then we can inductively define a sequence $(K_n)_{n \geq 1}$ of disjoint compact sets such that $\mu^\bullet(\chi_{K_n}|f|) > \alpha$ which is contradictory to the above lemma. For $K = \emptyset$ we have $\mu^\bullet(|f|) > \alpha$. Assume $K_1, \ldots, K_{n-1}$ constructed. Let ω be a relatively compact open neighborhood of $\bigcup_{i=1}^{n-1} K_i$. Then $\mu^\bullet(|f - \chi_\omega f|) \geq \mu^\bullet(|f - \chi_{\bar{\omega}} f|) > \alpha$ and there exists a compact set K such that $\mu^\bullet(\chi_K|f - \chi_\omega f|) > \alpha$. Then $K_n = K \cap \complement\omega$ satisfies the requirement. Thus the theorem is proved.

<u>Corollary</u> Let $T = I$ be discrete, let $\mu : \mathcal{K}(I) \longrightarrow E$ be a Radon measure, and let $x_i = \mu(\{i\})$, whence $\mu(\varphi) = \sum_{i \in I} \varphi(i)\, x_i$ for $\varphi \in \mathcal{K}(T)$. Then $f \in \mathcal{L}^1(\mu)$ iff $(f(i)\, x_i)_{i \in I}$ is B-summable, and in that case

$$\int gf d\mu = \sum_i g(i) f(i)\, x_i \quad \text{where} \quad g \in \mathcal{E}^\infty(I).$$

<u>Proof</u> By 1.6, if $f \in \mathcal{L}^1(\mu)$ $f = \lim_K \chi_K f$ in $\mathcal{L}^1$. Hence $\int f d\mu = \lim_K \int_K f d\mu = \lim_K \sum_{i \in K} f(i)\, x_i$. Replacing f by gf it follows that f is B-summable. The sufficiency results from the above theorem.

<u>Remark</u> In the above theorem the class of compact sets K such that $f/_K$ is continuous may be replaced by an smaller directed class $\mathcal{C}$, provided we still have the following: for every compact set K and every $\varepsilon > 0$ there is $K' \in \mathcal{C}$, $K' \subset K$, such that $\mu^\bullet(K - K') \leq \varepsilon$, or

equivalently $K = N + \sum_n K_n \quad K_n \in \mathcal{C}$. The proof is the same as before. As an application of this let us prove the following:

2.14 Theorem Let (f_n) be a sequence of μ-integrable functions converging μ.a.e to a function f, and assume $\lim_{n\to\infty} \int gf_n d\mu$ exists for every bounded Borel function. Then f is μ-integrable and $\lim_{n\to\infty} \int gf_n d\mu = \int gfd\mu$.

Proof We apply the above criterion with for $\mathcal{C}$ the class of compact sets K such that $f_{n/K}$ is continuous for all n, and converges uniformly to f/K. Let $(\lambda_p)_p$ be a bounded sequence A a subset of N, and put $u_n(A) = \sum_{p\in A} \lambda_p \int_{K_p} f_n d\mu$, where (K_p) is a sequence of disjoint compacts belonging to $\mathcal{C}$. By hypothesis $u(A) = \lim_{n\to\infty} u_n(A)$ exists for every A. It follows that u is also a countably additive set function [(1)], and since $\int_{K_p} f_n d\mu$ tends to $\int_{K_p} fd\mu$ it follows that the sequence $(\lambda_p \int_{K_p} fd\mu)_p$ is summable, whence $f \in \mathcal{L}^1(\mu)$. Now the maps $g \longrightarrow \int gf_n d\mu$ are equi-continuous on the bounded Borel functions with the uniform norm, by the Banach Steinhauss theorem, and for g equal to a finite sum $\sum \alpha_i \chi_{K_i}$ with $K_i \in \mathcal{C}$ $\lim_n \int gf_n d\mu = \int gfd\mu$. It follows from the density of these simple functions that the same relation holds for all g.

Remark When E is locally pseudo-convex it should be sufficient to assume $\lim_{n\to\infty} \int_A f_n d\mu$ exists for all Borel sets. (In the locally convex case it suffices to have it with all open sets A).

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(1) See Dunford and Schwartz [4] III 7.1. The proof of this theorem can be applied without modification, taking ν the measure of mass $1/2^n$ at the point $n \in N$.

Finally we prove some facts which will only be used in connection with the Fatou property in §5.

2.15 Theorem a) Let A be a μ-measurable set contained in a countable union of open μ-integrable sets. Then for every $\varepsilon > 0$ there exists an open set $\omega \supset A$ such that $\mu^\bullet(\omega - A) \leq \varepsilon$.

b) Let $f : T \longrightarrow [0, +\infty]$ be a μ-measurable function, zero in the complement of a countable union of open μ-integrable sets. Then for every $\varepsilon > 0$ there exists a lower semi-continuous function $g \geq f$ such that $\mu^\bullet(g-f) \leq \varepsilon$ (here $g(t) - f(t)$ is taken to be 0 when both $g(t)$ and $f(t)$ are equal to $+\infty$).

Proof. a) Assume first $A \subset 0, \chi_0 \in \mathcal{L}^1(\mu)$. Then by 2.5 there exists a compact set $K \subset 0 - A$ with $\mu^\bullet(0-A-K) \leq \varepsilon$. Consequently $\omega = 0 - K \supset A$ and $\mu^\bullet(\omega - A) \leq \varepsilon$. In the general case $A = \bigcup_n A_n$ where A_n satisfies the previous condition, and if $\omega_n \supset A_n$ with $\mu^\bullet(\omega_n - A_n) \leq \varepsilon/2n$ $\omega = \bigcup_n \omega_n \supset A$ and $\mu^\bullet(\omega - A) \leq \varepsilon$. b) Assume first that $f \leq n\chi_0$ where n is some number and $\chi_0 \in \mathcal{L}^1$. Then there exists a simple function $h = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$ with $f \leq h \leq n\chi_0$, $h - f \leq \delta\chi_0$ and consequently $\mu^\bullet(h-f) \leq \varepsilon/2$ for δ sufficiently small. Now by a) one can replace h by $g = \sum_{i=1}^{k} \alpha_i \chi_{\omega_i}$, with open $\omega_i \supset A_i$, such that $\mu^\bullet(g-h) \leq \varepsilon/2$, whence $\mu^\bullet(g-f) \leq \varepsilon$. In the general case f may be written $f(t) = \sum_n f_n(t)$ where f_n satisfies the previous condition and if $g_n \geq f_n$ g_n $\ell.s.c$ and bounded with $\mu^\bullet(g_n - f_n) \leq \varepsilon/2n$ we have $g(t) - f(t) \leq \sum_{n \geq 1} g_n(t) - f_n(t)$ (equality except possibly if $g(t)$ and $f(t)$ are both infinite) whence $\mu^\bullet(g-f) \leq \varepsilon$.

2.16 Theorem Let $(f_n)_{n \geq 1}$ be a sequence of μ-measurable functions with values in $[0, +\infty]$, such that $f_n \leq f_{n+1}$ and $f = \sup_n f_n$. Then $\mu^{\bullet}(f) = \sup_n \mu^{\bullet}(f_n)$.

Proof Assume first that f vanishes in the complement of a compact set. By 2.15 there exists $g \geq f$, lower semi-continuous such that $\mu^{\bullet}(g-f) \leq \varepsilon$. Let $\lambda < \mu^{\bullet}(f)$ and take $\varepsilon > 0$ such that $\lambda + \varepsilon < \mu^{\bullet}(f)$. Thus $\lambda + \varepsilon < \mu^{\bullet}(g)$ and there exists $\varphi \in \mathcal{K}_+$ such that $0 \leq \varphi \leq g$ and $\lambda + \varepsilon < \mu^{\bullet}(\varphi)$. Let $h = \inf(\varphi, f)$. Then $\varphi + f = h + \sup(\varphi, f) \leq h + g$, whence $\varphi - h \leq g - f$ (if $f(t) = +\infty$ $h(t) = \varphi(t)$) consequently $\mu^{\bullet}(\varphi - h) \leq \varepsilon$ and $\lambda + \varepsilon < \mu^{\bullet}(\varphi) \leq \mu^{\bullet}(h) + \varepsilon$ whence $\lambda < \mu^{\bullet}(h)$. Now put $h_n = \inf(h, f_n)$. Then $h_n \leq h_{n+1}$ and $h = \sup_n h_n$. By the dominated convergence theorem $\mu^{\bullet}(h-h_n)$ tends to zero, a fortiori $\mu^{\bullet}(h_n)$ tends to $\mu^{\bullet}(h)$ whence $\lambda < \mu^{\bullet}(h_n) \leq \mu^{\bullet}(f_n) \leq \mu^{\bullet}(f)$ for n sufficiently large. In the general case if f does not vanish in the complement of a compact set, there exists a compact K such that $\lambda < \mu^{\bullet}(\chi_K f)$ (by the definition of $\mu^{\bullet}$), then by the preceding argument it follows that $\lambda < \mu^{\bullet}(\chi_K f_n) \leq \mu^{\bullet}(f_n) \leq \mu^{\bullet}(f)$ for sufficiently large n. This ends the proof.

Corollary Let $f : T \longrightarrow [0, +\infty]$ be a μ-measurable function. Then $\mu^{\bullet}(f) = \sup_{|g| \leq f} \left| \int g d\mu \right|$ where $g = \sum_i \alpha_i \chi_{A_i}$ is a simple function (in which one may take A_i compact).

Proof By the preceding theorem it suffices to prove this when f is bounded and has compact support, whence $f \in \mathcal{L}^1(\mu)$. In that case if $\lambda < \mu^{\bullet}(f)$ there exists $\varphi \in \mathcal{K}$ with $|\varphi| \leq 1$ such that $\lambda < \left| \int \varphi f d\mu \right|$ (1.4d). Now there exists a sequence $(g_n)_{n \geq 1}$

of μ-measurable simple functions such that $g_n(t) \longrightarrow \varphi(t)f(t)$ and $|g_n| \leq f$. By the dominated convergence theorem $\int g_n d\mu$ tends to $\int f d\mu$, whence

$$\lambda < |\int g d\mu| = |\Sigma\, \alpha_i\, \mu(A_i)| \qquad \alpha_i \neq 0$$

for some simple function g with $|g| \leq f$. Finally by 2.5 the sets A_i may be replaced by compact sets $K_i \subset A_i$ without modifying the desired conditions, which proves the corollary.

Thus it is seen that many of the properties of the elementary integration theory for real or complex Radon measures are shared by Radon measures with values in complete metric topological vector spaces. There is an important difference however. Even when E is a Banach space and μ is an E valued Radon measure there may exist finite and even bounded μ-measurable functions f such that $\mu^\bullet(|f|) < +\infty$ and such that f is not μ-integrable. (Thus if T is discrete the injection $\mathcal{K}(T) \hookrightarrow \mathcal{C}_0(T)$ is a Radon measure $\mu^\bullet(f) = \sup_{t \in T} f(t)$, but $1 \notin \mathcal{L}^1(\mu)$). Nevertheless we shall prove later (§4) that if $E = L^p$ $0 < p < +\infty$ every μ-measurable function such that $\mu^\bullet(|f|) < +\infty$ is μ-integrable. Furthermore every Radon map $\mu : \mathcal{K}(T) \longrightarrow L^p$ is a Radon measure.

(1) See also [14] p. 122.

§3 Characteristic properties of Radon measures. "Weak compactness".

We continue to consider the case here where E is a complete metric topological vector space.

3.1 <u>Theorem</u> Let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon map. Then the following conditions are equivalent:

1) $\mathcal{L}^1(\mu)$ contains all bounded Borel functions with compact support (i.e. μ is a Radon measure).
2) $\mathcal{L}^1(\mu)$ contains each χ_K, where K is a compact G_δ.
3) For every sequence $(\varphi_n)_{n\geq 1}$ $\varphi_n \in \mathcal{K}(T)$ such that $\sum\limits_{n\geq 1} |\varphi_n(t)| \leq \chi_K$ for some compact set K, $\lim\limits_{n\to\infty} \mu(\varphi_n) = 0$.
4) For every bounded lower semi-continuous function with compact support, and every $\varepsilon > 0$, there exists $\varphi \in \mathcal{K}(T)$ such that $0 \leq \varphi \leq f$ and $\bar{\mu}(f-\varphi) \leq \varepsilon$.

<u>Proof</u> 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1). The first implication is obvious. For the second we prove under hypothesis 2).

<u>Lemma</u> Let $(\omega_n)_{n\geq 1}$ be a sequence of open K_σ's such that $\omega_n \supset \omega_{n+1}$ and $\bigcap\limits_n \omega_n = \emptyset$. Then $\lim\limits_n \bar{\mu}(\omega_n) = 0$.

<u>Proof</u> If ω is an open K_σ and K is a compact G_δ such that $\omega \subset K$ $K - \omega$ is a compact G_δ. Hence $\chi_\omega = \chi_K - \chi_{K-\omega} \in \mathcal{L}^1(\mu)$. Then by Theorem 1.10 $\inf\limits_{\substack{0\leq\varphi\leq\chi_\omega \\ \operatorname{supp}\varphi\subset\omega}} \bar{\mu}(\chi_\omega - \varphi) = 0$ and a fortiori

$\inf\limits_{K\subset\omega} \bar{\mu}(\omega - K) = 0$, where K is compact. Thus for the given sequence we can find compact sets $K_n \subset \omega_n$ such that

$\mu^\bullet(\omega_n - K_n) \leq \varepsilon/2^n$. If $H_n = \bigcap_{i=1}^{n} K_i$, $\omega_n - H_n \subset \bigcup_{i=1}^{n} \omega_i - K_i$ whence $\mu^\bullet(\omega_n - H_n) \leq \varepsilon$, and since $H_n = \emptyset$ for some n, we have $\mu^\bullet(\omega_n) \leq \varepsilon$, which proves the lemma.

This now implies 3). Given $(\varphi_n)_{n\geq 1}$ with $\Sigma|\varphi_n(t)| \leq \chi_K(t)$. Let $R_n(t) = \sum_{i\geq n} |\varphi_i(t)|$. Then R_n is lower semi-continuous, hence $\omega_n = \{t : R_n(t) > \varepsilon\}$ is open and $K \supset \omega_n \supset \omega_{n+1}$ while $\bigcap_n \omega_n = \emptyset$. Furthermore ω_n is a K_σ, namely the union of the compact sets $K_m = \{t : \sum_{i=n}^{i=n+m} |\varphi_i(t)| \geq \varepsilon + \frac{1}{m}\}$. Thus

$\lim_n \mu^\bullet(\omega_n) = 0$. Consequently by Theorem 1.11 (or directly) $\lim \mu^\bullet(R_n) = 0$ and since $|\mu(\varphi_n)| \leq \mu^\bullet(|\varphi_n|) \leq \mu^\bullet(R_n)$ the conclusion follows.

We next prove 3) $\Rightarrow$ 4). Let $f \geq 0$ be as given, and assume on the contrary that $\mu^\bullet(f - \varphi) > \alpha > 0$ for all $\varphi \in \mathcal{K}(T)$ with $0 \leq \varphi \leq f$. Then we can inductively define a sequence $(\varphi_i)_{i\geq 1}$ such that $\sum_{i=1}^{n} |\varphi_i(t)| \leq f(t)$ and $|\mu(\varphi_i)| > \alpha$. Indeed if the construction is done up to the rank $n-1$, $\mu^\bullet(f - \sum_{i=1}^{n-1} |\varphi_i|) > \alpha$, and consequently there exists $\varphi_{n+1} \in \mathcal{K}$ such that $|\varphi_{n+1}| \leq f - \sum_{i=1}^{n-1} |\varphi_i|$ and $|\mu(\varphi_{n+1})| > \alpha$. But from 3) it follows that $\lim_n \mu(\varphi_n) = 0$. This contradiction proves 4), which implies $f \in \mathcal{L}^1(\mu)$. In particular $\chi_\omega \in \mathcal{L}^1(\mu)$ for all relatively compact open sets. Now if K is an arbitrary compact K has a relatively compact open neighborhood ω, and $\chi_K = \chi_\omega - \chi_{\omega - K} \in \mathcal{L}^1(\mu)$, whence μ is a Radon measure by Theorem 2.1.

3.2 <u>Corollary</u> When $\mu^\bullet$ is additive on the lower semi-continuous functions μ is a Radon measure. This is the case when $E = \mathbb{C}$ or or $\mathbb{R}$ with the usual absolute value.

Proof The first assertion follows from Theorem 1.2 and condition 4) in the above theorem, since $\mu^{\bullet}(f-\varphi) = \mu^{\bullet}(f) - \mu^{\bullet}(\varphi)$. Assume $E = \mathbb{C}$, and let f_1 and f_2 be lower semi-continuous. Let $\lambda_1 < \mu^{\bullet}(f_1)$ $\lambda_2 < \mu^{\bullet}(f_2)$, and choose $\varphi_i \in \mathcal{K}(T)$ such that $\lambda_i < |\mu(\varphi_i)|$; multiplying φ_1 and φ_2 by a constant of unit modulus if necessary, this can be done such that $|\mu(\varphi_i)| = \mu(\varphi_i)$. Then $\lambda_1 + \lambda_2 < \mu(\varphi_1) + \mu(\varphi_2) = \mu(\varphi_1+\varphi_2) = |\mu(\varphi_1+\varphi_2)| \leq \mu^{\bullet}(f_1+f_2)$ since $|\varphi_1 + \varphi_2| \leq |\varphi_1| + |\varphi_2| \leq f_1 + f_2$. Thus λ_1 and λ_2 being arbitrary $\mu^{\bullet}(f_1) + \mu^{\bullet}(f_2) \leq \mu^{\bullet}(f_1+f_2)$ which together with 1.2 b proves the additivity [(1)].

Using condition 3) we shall later characterise the spaces E such that an arbitrary E-valued Radon map is a Radon measure.

We now consider a Radon map $\mu : \mathcal{K}(T) \longrightarrow E$ which is bounded (i.e. $\{\mu(\varphi)\}_{|\varphi| \leq 1}$ is bounded in E), or equivalently which is continuous for the topology of uniform convergence in T: For every $\varepsilon > 0$ there is $\delta > 0$ such that $|\varphi| \leq \delta$ implies $|\mu(\varphi)| \leq \varepsilon$, and consequently $\mu^{\bullet}(\delta) \leq \varepsilon$. Then μ has a continuous extension to the space $\mathcal{C}_0(T)$ of continuous functions tending to ..

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(1) One thus defines a positive Radon measure $|\mu|$, total variation of μ, by putting $|\mu|(\varphi) = \mu^{\bullet}(\varphi)$ for $\varphi \geq 0$, and by 1.2, $|\mu|^{\bullet} = \mu^{\bullet}$, whence $\mathcal{L}^1(|\mu|) = \mathcal{L}^1(\mu)$, and $\int |f| d|\mu| \leq |\mu|^{\bullet}(|f|) = \mu^{\bullet}(|f|)$ with equality for $f \in \mathcal{K}$ and consequently everywhere: $\mu^{\bullet}(|f|) = \int |f| d|\mu|$ for all $f \in \mathcal{L}^1$. By 1.4 d) $\int |f| d|\mu| = \sup_{|\varphi| \leq 1} |\int \varphi f d\mu|$.

zero at infinity. More precisely we then have $\mathcal{C}_o(T) \hookrightarrow \mathcal{L}^1(\mu)$, for if $f \in \mathcal{C}_o(T)$ and $\varphi \in \mathcal{K}(T)$ with $|f-\varphi| \leq \delta$ it follows that $\dot{\mu}(|f-\varphi|) \leq \dot{\mu}(\delta) \leq \varepsilon$, and if $|f| \leq \delta$ $\dot{\mu}(|f|) \leq \varepsilon$. Thus the restriction to $\mathcal{C}_o(T)$ of the integral $f \longrightarrow \int f d\mu$ coincides with the continuous extension of μ to $\mathcal{C}_o(T)$.

3.3 Theorem Let $\mu : \mathcal{K}(T) \longrightarrow E$ be a bounded Radon map. Then the following conditions are equivalent:

1) $\mathcal{L}^1(\mu)$ contains all bounded Borel functions.

2) μ is a Radon measure and $1 \in \mathcal{L}^1(\mu)$.

3) For every bounded lower semi-continuous function $f \geq 0$, and $\varepsilon > 0$, there exists $\varphi \in \mathcal{K}$ with $0 \leq \varphi \leq f$ and $\dot{\mu}(f-\varphi) \leq \varepsilon$.

4) a) For every compact set K and $\varepsilon > 0$ there exists an open set $\omega \supset K$ such that $\dot{\mu}(\omega - K) \leq \varepsilon$, and

   b) For every $\varepsilon > 0$ there exists a compact set K such that $\dot{\mu}(T-K) \leq \varepsilon$.

5) For every open set ω there exists $K \subset \omega$, compact, such that $\dot{\mu}(\omega - K) \leq \varepsilon$.

6) For every sequence $(\omega_n)_{n \geq 1}$ of open sets such that $\omega_n \supset \omega_{n+1}$ and $\bigcap_n \omega_n = \emptyset$ $\lim_n \dot{\mu}(\omega_n) = 0$.

7) For every sequence $(\omega_n)_{n \geq 1}$ of disjoint open sets $\lim_n \dot{\mu}(\omega_n) = 0$.

8) For every sequence $(K_n)_{n \geq 1}$ of disjoint compact sets $\lim_n \dot{\mu}(K_n) = 0$.

9) For every sequence $(\varphi_n)_{n \geq 1}$, $\varphi_n \in \mathcal{K}(T)$, such that $\sum_n |\varphi_n(t)| \leq 1$, $\lim_n \mu(\varphi_n) = 0$.

10) μ transforms weakly compact subsets of $\mathcal{C}_o(T)$ into relatively compact subsets of E .

Proof For the moment we leave aside condition 10) which is different from the others in that it only involves the Banach space structure of $\mathcal{C}_o(T)$. The equivalence of the other conditions will be proved as follows:

$$\begin{tikzcd}[column sep=small]
 & 8) \arrow[r, Rightarrow] & 7) \arrow[r, Rightarrow] & 5) \arrow[r, Rightarrow] & 4) \arrow[r, Rightarrow] & 2) \arrow[dr, Rightarrow] & \\
1) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & & & & 1) \\
 & 6) \arrow[r, Rightarrow] & 9) \arrow[r, Rightarrow] & 3) \arrow[rr, Rightarrow] & & 2) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Top line: 1) implies 8) by the dominated convergence theorem.
8) $\Longrightarrow$ 7) by proposition 1.7.
7) $\Longrightarrow$ 5). Assume $\mu^*(\omega - K) > \alpha$ for all compact sets $K \subset \omega$. Then one can inductively define a sequence of disjoint open sets ω_n , whose closure is compact and contained in ω , such that $\mu^*(\omega_n) > \alpha$ in contradiction with 7. Assume the construction done up to rank $n-1$, and let $K = \bigcup_{i=1}^{n-1} \bar{\omega}_i$. Then $K \subset \omega$ and K is compact, whence $\mu^*(\omega - K) > \alpha$, and by 1.7 there exists a compact set $H \subset \omega - K$ such that $\mu^*(H) > \alpha$. Let ω_n be an open neighborhood of H with compact closure such that $H \subset \omega_n \subset \bar{\omega}_n \subset \omega - K$. Then $\mu^*(\omega_n) > \alpha$ and the sequence $(\omega_i)_{i=1-n}$ satisfies the induction hypothesis.
5) $\Longrightarrow$ 4) b) is obvious. For a) let O be an open neighborhood of K ; then there exists a compact $H \subset O - K$ such that $\mu^*(O-K-H) \leq \varepsilon$ whence $\mu^*(\omega - K) \leq \varepsilon$ where $\omega = O - H$.
4) $\Longrightarrow$ 2) For $\omega \supset K$ there exists $\varphi \in \mathcal{K}$ with $\chi_K \leq \varphi \leq \chi_\omega$,

whence by a) $\mu^{\bullet}(\varphi - \chi_K) \leq \varepsilon$ and consequently $\chi_K \in \mathcal{L}^1(\mu)$. Thus a) implies that μ is a Radon measure and similarly b) implies $1 \in \mathcal{L}^1(\mu)$.

2) $\Longrightarrow$ 1) by Theorem 1.9.

Bottom line: 1) implies 6) by the dominated convergence theorem.

6) $\Longrightarrow$ 9) $\Longrightarrow$ 3) The proof is entirely similar to the proof of Theorem 3.1.

3) $\Longrightarrow$ 2) For μ is a Radon measure by Theorem 3.1 and applying condition 3) with $f = 1$ it is obvious that $1 \in \mathcal{L}^1$.

This ends the proof of the equivalence of the first nine conditions.

3.4 Definition We shall say that a Radon map which satisfies the above conditions possesses the global extension property (as opposed to Radon measures which have the local extension property).

If T is infinite discrete the natural injection $\mu : \mathcal{K}(T) \longrightarrow \mathcal{C}_0(T)$ has the local but not the global extension property, since $\mathcal{L}^1(\mu) = \mathcal{C}_0(T)$ and $1 \notin \mathcal{L}^1(\mu)$.

If $\mu : \mathcal{C}_0(T) \longrightarrow \mathbb{C}$ is a continuous linear form it again follows from the additivity of $\mu^{\bullet}$ on the l.s.c. functions and condition 3 or 7 in Theorem 3.3, that μ has the global extension property (which is of course well known). In particular it follows from the dominated convergence theorem that a sequence $(f_n)_{n \geq 1}$ of function $f_n \in \mathcal{C}_0(T)$ converges weakly to $f \in \mathcal{C}_0(T)$, if and only if $(f_n)_{n \geq 1}$ is uniformly bounded and $f(t) = \lim_n f_n(t)$ for all t.

Let us now prove the equivalence of condition 10) with the others: 1) implies 10): let $H \subset \mathcal{C}_0(T)$ be weakly compact. To show that $\mu(H)$ is relatively compact it suffices, E being metrisable, to prove that any sequence $(f_n)_{n \geq 1}$ extracted from H contains a subsequence $(f_{n_K})_{K \geq 1}$ such that $\mu(f_{n_K}) = \int f_{n_K} d\mu$ converges in E . Now by the theorem of Eberlein-Smulian there exists a subsequence (f_{n_K}) which converges weakly to some $f \in H$. Thus $|f_{n_K}(t)| \leq M$ and $f_{n_K}(t) \longrightarrow f(t)$ for all t, and by the dominated convergence theorem applied to μ, $\mu(f_{n_K}) = \int f_{n_K} d\mu$ tends to $\int f d\mu$. Conversely 10) implies 9). Let $\sum_n |\varphi_n(t)| \leq 1$. Then $\varphi_n(t)$ tends to zero, consequently φ_n tends weakly to zero in $\mathcal{C}_o(T)$ by the above remarks. Thus by hypothesis the sequence $\{\mu(\varphi_n)\}$ is relatively compact in E. To prove that it tends to zero it suffices to prove that for any subsequence $(\varphi_{n_K})_K$ such that $\mu(\varphi_{n_K})$ converges, $\lim_K \mu(\varphi_{n_K}) = 0$. Assume then that $x = \lim_K \mu(\varphi_{n_K})$, and extracting if necessary another subsequence of $(\varphi_{n_K})_K$ assume $\sum_K |x - \mu(\varphi_{n_K})| < +\infty$[1]. Then E being complete the sums $\sum_{K=1}^{n} x - \mu(\varphi_{n_K})$ converges as n tends to infinity, that is $\lim_{n \to \infty} nx - \mu(\psi_n) = y$ exists, where $\psi_n = \sum_{K=1}^{n} \varphi_{n_K}$. Thus $\lim_{n \to \infty} x - \mu(\frac{1}{n} \psi_n) = 0$,

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(1) The use of series in this connection was suggested by Ph. Turpin.

i.e. $x = \lim_n \mu(\frac{1}{n} \Psi_n)$. But $|\Psi_n(t)| \leq 1$, hence $\frac{1}{n} \Psi_n$ tends to zero uniformly and consequently $x = 0$. This ends the proof of Theorem 3.3.

<u>Remark</u> It is known [(1)] that when E is locally convex condition 10) is equivalent to:

11) μ maps bounded subsets into weakly relatively compact subsets, i.e. μ is weakly compact.

Furthermore if $E = \ell^\infty(I)$ the space of bounded scalar families $(\lambda_i)_{i \in I}$ with the norm $\sup_{i \in I} |\lambda_i|$, a bounded Radon measure $\mu : \mathcal{C}_o(T) \longrightarrow E$ is of the form $\mu(\varphi) = (\mu_i(\varphi))_{i \in I}$ where $(\mu_i)_{i \in I}$ is a uniformly bounded family of complex Radon measures. We then have $|\mu(\varphi)| = \sup_{i \in I} |\mu_i(\varphi)|$ and $\mu^\sim(\omega) = \sup_{i \in I} |\mu_i|(\omega)$ when ω is an open set, and the weak

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(1) See A. Grothendieck [5], Theorem 4, p. 153 and preceding remark.

compactness of the map μ is equivalent to the fact that the family $(\mu_i)_{i \in I}$ is weakly relatively compact in $M(T) = \mathcal{C}_0(T)'$ (the weak topology being $\sigma(M,M')$). In that sense it can be said that the conditions in Theorem 3.3 are generalizations of known weak compactness criteria (every Banach space is a closed subspace of some space $\ell^\infty(I)$).

Condition 6) is by the dominated convergence theorem equivalent to the slightly stronger requirement:

6') For any sequence $(A_n)_{n \geq 1}$ of Borel sets with $A_n \supset A_{n+1}$ and $\bigcap_n A_n = \emptyset$ $\lim_n \mu^\cdot(A_n) = 0$. If $\mu = (\mu_i)_{i \in I}$ this would be called uniform countable additivity for the family $(\mu_i)_{i \in I}$ (or rather $(|\mu_i|)_{i \in I}$).

There is another weak compactness criterion, due to Bartle, Dunford and Schwartz (and which is the basis of their theory of abstract vector measures): namely "uniform absolute continuity". In our case clearly if there exists a bounded positive measure λ such that $\lim_{\lambda(A) \to 0} \mu^\cdot(A) = 0$ (where A is a Borel set) the conditions in Theorem 3.3 are satisfied. Now when μ has the global extension property (i.e. is a Radon measure for which $1 \in \mathcal{L}^1(\mu)$) it is easy to see in the usual way that the following conditions are equivalent, A being a Borel set:

a) $\lim_{\lambda(A) \to 0} \mu^\cdot(A) = 0$

b) $\lambda(A) = 0$ implies $\mu^\cdot(A) = 0$

c) $\lambda(K) = 0$, K compact, implies $\mu(K) = 0$.

When E is locally convex it follows from the work of Bartle, Dunford and Schwartz that there exists a bounded positive measure λ

such that $\lambda(A) = 0$ is equivalent to $\mu^{\cdot}(A) = 0$. In other terms: the Boolean σ - algebra of Borel sets modulo the μ-negligible sets (with the metric $\mu^{\cdot}(A \Delta B)$ for which it is complete) is a measure algebra. When E is not locally convex we do not know whether this is the case even when $T = [0, 1]$. If such a measure λ were known to exist certain of the theorems in §2 could be deduced from the corresponding theorems for ordinary measures (e.g. Theorem 2.8 could be deduced from Egoroff's theorem).

We now give an example which shows that condition 2) in Theorem 3.1 cannot be much weakened.

3.5 Example of a Radon map $\mu : \mathcal{C}[0, 1] \longrightarrow E$ where E is a Banach space, and such that $\mathcal{L}^1(\mu)$ contains all step functions, and consequently their uniform limits the ruled functions, but such that μ is not a Radon measure.

Construction: Let $F : [0, 1] \longrightarrow [0, 1]$ be the well known continuous non-decreasing singular function such that $F(0) = 0$ $F(1) = 1$ and $F'(t) = 0$ a.e, and let $(F_n)_{n \geq 1}$ be a sequence of absolutely continuous, non-decreasing functions converging uniformly to F (for example the functions usually used to construct F as a limit). Then for every continuous function $\varphi \in \mathcal{C}[0, 1]$, $\lim_{n \to \infty} \int \varphi \, d F_n = \int \varphi \, dF$. We take $E = c(N)$ the space of convergent sequences $x = (\lambda_n)_{n \geq 1}$ with the norm $|x| = \sup_n |\lambda_n|$ and we put $\mu(\varphi) = \{ \int \varphi \, d F_n \}_{n \geq 1} = \{ \int \varphi \, f_n dt \}_{n \geq 1}$, if $F_n(x) = \int_0^x f_n(t) dt$ which we may assume. Then if ω is an open set

$$\mu^{\bullet}(\omega) = \sup_{|\varphi| \leq \chi_\omega} |\mu(\varphi)| = \sup_n \sup_{|\varphi| \leq \chi_\omega} \left|\int \varphi f_n dt\right| = \sup_n \int_\omega f_n dt$$

In particular if ω is an open interval (a,b) or $[0,b)$ or $(a,1]$,

$$\mu^{\bullet}(\omega) = \sup_n F_n(b) - F_n(a)$$

Now since F_n converges uniformly the sequence (F_n) is equi-continuous (but not equi-absolutely continuous which is the point of the construction) and consequently for every $\varepsilon > 0$ there exists δ such that $|b-a| \leq \delta$ implies $\mu^{\bullet}(a,b) \leq \varepsilon$. In particular $\mu^{\bullet}(\{c\}) = 0$ for any $c \in [0, 1]$. Now let I be any interval with extremities a, b such that $0 \leq a < b \leq 1$. Then we can find a', a'', b', b'' with $a' \leq a < a'' < b'' < b \leq b'$ (and the equalities holding only if $a = 0$ or $b = 1$) such that $I' = (a', b')$ is open (in $[0, 1]$) $I'' = [a'', b'']$ is closed and such that $\mu^{\bullet}(I' - I'') \leq \mu^{\bullet}(a', a'') + \mu^{\bullet}(b'', b') \leq \varepsilon$, and since $I'' \subset I \subset I'$ a fortiori $\mu^{\bullet}(I - I'') \leq \varepsilon$. Since there exists a continuous function φ such that $\chi_{I''} \leq \varphi \leq \chi_I$ it follows that $\mu^{\bullet}(\chi_I - \varphi) \leq \varepsilon$, consequently $\chi_I \in \mathcal{L}^1(\mu)$. Thus $\mathcal{L}^1(\mu)$ contains all step functions (linear combinations of characteristic functions of intervals) and their uniform limits the ruled functions. It remains to show that μ is not a Radon measure. Let $\tilde{\mu}(A) = \{\int_A f_n dt\}_{n \geq 1}$ when A is a Borel set and assume $\tilde{\mu}(A) \in c(N)$ for all Borel sets. Then by the theorem of Vitali-Hahn-Saks, for every $\varepsilon > 0$ there is δ such that measure $(A) \leq \delta$ implies $\int_A f_n dt \leq \varepsilon$ for all n, in particular for $0 \leq s_1 < t_1 \leq s_2 < t_2 \ldots s_n < t_n \leq 1$, $\sum_i |t_i - s_i| \leq \delta$ implies $\sum_i F_n(t_i) - F_n(s_i) \leq \varepsilon$ for all n, whence $\sum_i F(t_i) - F(s_i) \leq \varepsilon$. But this is contradictory because F is not absolutely continuous. Thus we do not have $\tilde{\mu}(A) \in \mathbb{C}$. A fortiori μ is not a Radon
$c(N)$

measure (cf. Th. 4.1a). The above example is also an example of a case where we have a continuous function $t \longrightarrow x(t) \in E$ such that for all $\varphi \in \mathcal{C}[0, 1]$ the Stieltjes integral

$$\mu(\varphi) = \int \varphi \, dx = \lim \sum_{i=1}^{n} \varphi(\xi_i)x(t_i) - x(t_{i-1})$$

$$0 = t_o < \xi_1 < t_1 < \xi_2 < t_2 \; . \; . < \xi_n < t_n = 1 \; \lim \sup|t_i - t_{i-1}| = 0)$$

exists but where μ is not a Radon measure. However we have the following:

3.6 Proposition Let $m : I \longrightarrow m(I)$ be an additive function defined on the bounded intervals of $\mathbb{R}$, and for the step function $f = \sum_{i=1}^{n} \alpha_i \chi_{I_i}$ put $m(f) = \sum_{i=1}^{n} \alpha_i \, m(I_i)$. Then there exists a Radon measure $\mu : \mathcal{K}(\mathbb{R}) \longrightarrow E$ such that $m(I) = \mu(I)$ if and only if m has the following property: $\lim_n f_n(t) = 0$, $|f_n(t)| \leq 1$, and $f_n(t) = 0$ in the complement of a bounded interval, implies $\lim_n m(f_n) = 0$.

Proof Obviously the condition is necessary by the dominated convergence theorem since $m(I) = \mu(I)$ implies $m(f) = \int f \, d\mu$. Conversely the condition first implies that m may be continuously extended to the ruled functions with compact support and the given convergence property of m then clearly extends to the ruled functions also. Thus if we put $\mu(\varphi) = m(\varphi)$ for $\varphi \in \mathcal{K}(T)$ μ is a Radon measure by Theorem 3.1 property 3). It remains to show that $m(I) = \mu(I)$. Given a bounded interval I there exists a sequence of continuous functions $(\varphi_n)_{n \geq 1}$ such that $\lim_n \varphi_n(t) = \chi_I(t)$, $|\varphi_n(t)| \leq 1$ and $\operatorname{supp} \varphi_n$ contained in a neighborhood of I. Then $m(\varphi_n) = \mu(\varphi_n)$ tends to $m(I)$ by the hypothesis on m, and to $\mu(I)$ by the dominated convergence theorem.

3.7 Corollary Let m be an additive interval function and assume its extension to the step functions is continuous for the topology induced by some space $L^p(\nu)$, $(0 < p < +\infty)$ where ν is a positive Radon measure on $\mathbb{R}$. Then there is a Radon measure $\mu : \mathcal{K}(\mathbb{R}) \to E$ such that $m(I) = \mu(I)$.

3.8 Example $E = L^2(\Omega, P)$ (Ω, P) Probability space, and $t \to X(t)$ is standard Brownian motion. Put for $f = \Sigma \alpha_i \chi_{(t_{i-1}, t_i]}$ $m(f) = \Sigma \alpha_i (X(t_i) - X(t_{i-1}))$ then, because of the orthonality of the increments $|m(f)|_2^2 = \Sigma |\alpha_i|^2 (t_i - t_{i-1}) = \int |f|^2 dt$. Thus $\mu(\varphi) = \int \varphi \, dX$ exists and $|\mu(\varphi)|^2 = \int |\varphi|^2 dt$, whence for $f \geq 0$ $\ell.s.c.$ $\mu^*(f)^2 = \sup\limits_{|\varphi| \leq f} \int |\varphi|^2 dt = \int f^2 dt$ and consequently $\mathcal{L}^1(\mu) = \mathcal{L}^2(\nu)$ where ν is Lebesgue measure.

§4 Extension to arbitrary topological vector spaces.

Let E be a separated topological vector space [1]. A Radon map $\mu : \mathcal{K}(T) \longrightarrow E$ is by definition a linear map such that for every compact set $K \subset T$ the restriction of μ to the space $\mathcal{K}(T, K)$ of functions $\varphi \in \mathcal{K}(T)$ with support in K, is continuous with respect to uniform convergence. Let $x \longrightarrow |x|_d$ be a continuous invariant pseudo-metric on E ($|x+y|_d \leq |x|_d + |y|_d$, $|\lambda x|_d \leq |x|_d$ for $|\lambda| \leq 1$, $\lim\limits_{x \to 0} |x|_d = 0$), let E_d be the completion of the associated metric t.v.s. $E/\{x : |x|_d = 0\}$ and let $\pi_d : E \longrightarrow E_d$ be the quotient map. The topology in E is then the coarsest for which all the linear maps π_d are continuous.

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(1) It is not essential to assume E separated for the definition of $\mathcal{L}^1(\mu)$, but $\int f \, d\mu$ can be defined unambiguously only when E is separated.

We put $\mu_d = \pi_d \circ \mu$, so that μ_d is a Radon map with values in E_d. Then for $f \geq 0$ lower semi-continuous $\mu_d^{\bullet}(f) = \sup_{|\varphi| \leq f} |\mu(\varphi)|_d$. We define a function or set to be μ-integrable (resp. to possess the Lusin property with respect to μ, resp. to be μ-negligible) if it is μ_d - integrable (resp. possesses the Lusin property with respect to μ_d, resp. is μ_d - negligible) for every d. Thus $\mathcal{L}^1(\mu) = \bigcap_d \mathcal{L}^1(\mu_d)$. The topology in $\mathcal{L}^1(\mu)$ is by definition the coarsest for which the injections $\mathcal{L}^1(\mu) \hookrightarrow \mathcal{L}^1(\mu_d)$ are continuous, i.e. the one defined by the invariant pseudo-metrics

$$f \longrightarrow \mu_d^{\bullet}(|f|) .$$

Thus $\mathcal{L}^1(\mu)$ is a topological vector space. The associated Hausdorff space $L^1(\mu)$ is the quotient space $\mathcal{L}^1(\mu)/_{N(\mu)}$ where $N(\mu)$ is the subspace of functions equal to zero μ- almost everywhere. The space $L^1(\mu)$ is not in general complete or even quasi-complete [(1)].

Since $|\mu(\varphi)|_d \leq \mu_d^{\bullet}(|\varphi|)$, and $\mathcal{K}(T)$ is dense in $\mathcal{L}^1(\mu)$, the map μ has a unique continuous extension $f \longrightarrow \int f\, d\mu$ from $\mathcal{L}^1(\mu)$ to the completion $\hat{E}$ of E.

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(1) See remarks preceding Theorem 6.3.

By definition μ is a Radon measure when $\mathcal{L}^1(\mu)$ contains all bounded Borel functions with compact support, equivalently when μ_d is a Radon measure for all d. In that case the sets in the σ - algebra $\mathcal{B}^\mu = \bigcap_d \mathcal{B}^{\mu_d}$ are called <u>μ-measurable</u> and the $\mathcal{B}^\mu$-measurable functions, which are precisely the ones having the Lusin property with respect to μ, are also called μ-measurable. The following theorem summarises the essential properties and can be used instead of the above definitions:

4.1 <u>Theorem</u> a) E and F being separated topological vector spaces, let $u : E \longrightarrow F$ be a continuous linear map and put $\nu = u \circ \mu$ where μ is an E-valued Radon map. Then $\mathcal{L}^1(\mu) \hookrightarrow \mathcal{L}^1(\nu)$, the injection being continuous, and $\int f\, d\nu = u \int f\, d\mu$ for every $f \in \mathcal{L}^1(\mu)$ (u designating the continuous extension of the given linear map to the completions of E and F).

Every function with the Lusin property for μ has the Lusin property for ν.

If μ is a Radon measure so is ν and every μ-measurable function is ν-measurable. Finally every μ-negligible set is ν-negligible.

b) Assume the topology of E is the coarsest for which linear maps $u_i : E \longrightarrow E_i$ into topoligical vector spaces are continuous and put $\mu_i = u_i \circ \mu$. Then $\mathcal{L}^1(\mu) = \bigcap_i \mathcal{L}^1(\mu_i)$, the topology of $\mathcal{L}^1(\mu)$ being the coarsest for which the injections $\mathcal{L}^1(\mu) \hookrightarrow \mathcal{L}^1(\mu_i)$ are continuous. In particular μ is a Radon measure if and only if μ_i is a Radon measure for all i and in that case $\mathcal{B}^\mu = \bigcap_i \mathcal{B}^{\mu_i}$ and a function is μ-measurable iff it is μ_i - measurable for all i. Finally a set is μ-negligible iff it is μ_i - negligible for all i.

Proof For a) observe that if $y \longrightarrow |y|_\delta$ is a continuous invariant pseudo-metric on F, $x \longrightarrow |u(x)|_\delta = |x|_d$ is a continuous invariant pseudo-metric on E and for $f \geq 0$ l.s.c,

$$\nu^\bullet_\delta(f) = \sup_{|\varphi| \leq f} |\nu(\varphi)|_\delta = \sup_{|\varphi| \leq f} |\mu(\varphi)|_d = \mu^\bullet_d(f),$$

whence $\nu^\bullet_\delta = \mu^\bullet_d$.

For the proof of b) observe that if $|\ |_d$, $|\ |_{d_1} \cdots |\ |_{d_n}$ are continuous pseudo-metric on E such that $|x|d_i < \varepsilon_i$, $i = 1 — n$ implies $|x|_d \leq \varepsilon$, then $f \geq 0$ and $\mu^\bullet_{d_i}(f) < \varepsilon_i$ $i = 1 — n$ implies $\mu^\bullet_d(f) \leq \varepsilon$. Thus one may in the definition of $\mathcal{L}^1(\mu)$ etc, restrict oneself to a fundamental family of invariant pseudo-metrics and the verification of b) presents no difficulty. We leave the details of the proof to the reader.

Corollary If E is a locally convex space one can take the spaces E_i to be Banach spaces, in which case $L^1(\mu_i)$ is a Banach space, and $\mathcal{L}^1(\mu) = \bigcap_i \mathcal{L}^1(\mu_i)$ is locally convex. If furthermore E is metrisable, $L^1(\mu)$ is a Frechet space and its topology, as defined in §1, is the coarsest for which the canonical maps $L^1(\mu) \longrightarrow L^1(\mu_i)$ are continuous[1].

4.2 Theorem Let E be a quasi-complete separated topological vector space (in E closed bounded sets are complete). Then $\int f\,d\mu \in E$ for all $f \in \mathcal{L}^1(\mu)$.

Proof Let $\hat{E}$ be the completion of E. Then $\int f\,d\mu \in \hat{E}$ and it suffices to prove that $\int f\,d\mu$ is in the closure of a bounded subset of E. We may assume $f \geq 0$, and we first further assume that f is bounded and has compact support, e.g. $0 \leq f \leq \chi_K$. Then if H

(1) Thus the present definition coincides, when E is locally convex, with the definition of $\mathcal{L}^1(\mu)$ previously given in [14].

is a compact neighborhood of K, f is in $\mathcal{L}^1(\mu)$ in the closure of the set $\{\varphi \in \mathcal{K}(T) : |\varphi| \leq \chi_H\}$ and $\int f\,d\mu$ is in the closure of the set $\{\mu(\varphi)\}_{|\varphi| \leq \chi_H}$, whence $\int f\,d\mu \in E$. If f has compact support but is no longer bounded we put $f_n = \inf(f,n)$ and observe that $f = \lim_n f_n$ in $\mathcal{L}^1(\mu)$ (by Theorem 1.11; see also the proof of Theorem 1.9). Then $\int f\,d\mu = \lim_n \int f_n d\mu$, and a convergent sequence being bounded, $\int f\,d\mu \in E$. Finally in general this proves that $\int \varphi f\,d\mu \in E$ for all $\varphi \in \mathcal{K}(T)$. Now clearly f is in $\mathcal{L}^1(\mu)$ in the closure of the bounded set $\{\varphi f\}_{|\varphi| \leq 1}$, and consequently $\int f\,d\mu$ is in the closure of $\{\int \varphi f\,d\mu\}_{|\varphi| \leq 1}$, whence $\int f\,d\mu \in E$.

By applying Theorem 4.1 to the case where the spaces E_i are complete metric spaces some of the results obtained in the previous sections can be immediately generalised to the case of arbitrary topological vector spaces. We summarise them for later reference:

4.3 Theorem Let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon map, where E is a quasi-complete t.v.s.

1) Every $f \in \mathcal{L}^1(\mu)$ possesses the Lusin property and conversely if f possesses the Lusin property and $|f| \leq g$ with $g \in \mathcal{L}^1(\mu)$, it follows that $f \in \mathcal{L}^1(\mu)$.

2) If μ is a Radon measure f possesses the Lusin property iff f is measurable with respect to the σ - algebra $\mathcal{B}^\mu = \{A : \chi_{A \cap K} \in \mathcal{L}^1(\mu)\ \forall\ K$ compact$\}$, (in which case f is called μ-measurable). All Borel functions are μ-measurable.

3) The map μ is a Radon measure (resp. has the global extension

property) iff $\sum_n |\varphi_n(t)| \leq \chi_K(t)$ (resp. $\Sigma|\varphi_n(t)| \leq 1$) implies $\lim_n \mu(\varphi_n) = 0$.

4) If μ is a Radon measure the 'dominated convergence theorem' is valid: If $f_n \in \mathcal{L}^1(\mu)$, $f_n(t)$ tends to $f(t)$ μ a.e and $|f_n| \leq g$ where $g \in \mathcal{L}^1(\mu)$, f_n tends to f in $\mathcal{L}^1(\mu)$, which means that $\int hf_n d\mu$ tends to $\int hf\, d\mu$ in E, uniformly with respect to the μ-measurable functions h such that $|h| \leq 1$.

5) If μ is a Radon measure one has the following integrability criterion: A complex function f is μ-integrable iff f is μ-measurable and for every sequence $(K_n)_{n \geq 1}$ of disjoint compact sets such that $f/_{K_n}$ is continuous, the sequence $(\int_{K_n} f\, d\mu)_n$ is B-summable in E. When E is locally pseudo-convex it is sufficient that $\lim_n \int_{K_n} f\, d\mu = 0$.

6) Let $(x_i)_{i \in I}$ be a family of elements in E, and let $\mu : \mathcal{K}(I) \longrightarrow E$ be the discrete measure defined by $\mu(\varphi) = \sum_{i \in I} \varphi(i) x_i$. Then $f \in \mathcal{L}^1(\mu)$ iff $(f(i)x_i)_{i \in I}$ is B-summable and in that case $\int f d\mu = \sum_{i \in I} f(i) x_i$.

Concerning discrete measures we have:

4.4 Theorem Let E be a quasi-complete t.v.s., and let $\mu : \mathcal{K}(T) \dashrightarrow E$ be a discrete measure: $\mu(\varphi) = \sum_{i \in I} \varphi(i) x_i$. Then

1) μ is bounded (i.e. $\{\mu(\varphi)\}_{|\varphi| \leq 1}$ is bounded in E) iff $(f(i)x_i)_{i \in I}$ is summable for every $f = (f(i))_{i \in I} \in c_0(I)$.

2) When μ is bounded the following are equivalent:

a) μ has the global extension property (i.e $\ell^\infty(I) \subset \mathcal{L}^1(\mu)$, or $1 \in \mathcal{L}^1(\mu)$).

b) $(x_i)_{i \in I}$ is B-summable.

c) $(x_i)_{i \in I}$ is summable.

d) The map $\mu : c_o(I) \longrightarrow E$ is compact.

<u>Proof</u> Since E is a subspace of a product of complete metrisable spaces it suffices to consider the case where E is metrisable. Boundedness being equivalent to continuity 1) is easily established by reduction to the case where $I = N$ is countable and by using the Banach Steinhauss theorem. In 2) a) and b) are obviously equivalent by the last part of the preceding theorem. Condition d) implies a) by Theorem 3.3. Since a weakly compact subset of $c_o(I)$ is certainly bounded. Conversely a) implies d): Let $f_n \in c_o(I)$ with $|f_n(i)| \leq 1$. There exists a countable subset $J \subset I$ such that $f_n(i) = 0$ for all n, and $i \in \complement J$. Thus one can extract a subsequence f_{n_k} which converges pointwise, and consequently $\mu(f_{n_k}) = \int f_{n_k}\, d\mu$ converges in E by the dominated convergence theorem, which proves d). It remains to prove that c) implies b). Let $| \ |$ be a metric defining the topology in E. By the continuity there exists for every $\varepsilon > 0$ some $\delta > 0$ such that $\varphi \in \mathcal{K}(I)$ $|\varphi(i)| \leq \delta$ implies $|\sum_i \varphi(i) x_i| \leq \varepsilon$, in particular for $f \in \ell^\infty(I)$ and $K \subset I$ a finite set, $|f(i)| \leq \delta$ implies $|\sum_{i \in K} f(i) x_i| \leq \varepsilon$. Now by c) if $f' \in \ell^\infty$ takes only a finite number of values $(f'(i)x_i)_{i \in I}$ is summable. Choose f' such that $|f(i) - f'(i)| \leq \delta$ for all i. There exists a finite set K_o such that for $K \cap K_o = \emptyset$ $|\sum_{i \in K} f'(i) x_i| \leq \varepsilon$. Then $|\sum_{i \in K} f(i) x_i| \leq 3\varepsilon$, whence by Cauchy's criterion $(f(i)x_i)_{i \in I}$ is summable.

We now briefly describe a second definition of $\mathcal{L}^1(\mu)$ which is valid for arbitrary topological vector spaces and which involves the neighborhoods of zero in E directly rather than the invariant pseudo-metrics. However this will not be used anywhere except in Theorem 5.4, the more important corollary of which can be formulated and proved independently.

Let V be a neighborhood of zero in E which is closed and balanced ($\lambda V \subset V$ for $|\lambda| \leq 1$). Put $|x|_V = \inf\{\lambda > 0 : x \in \lambda V\}$. Then $V = \{x : |x|_V \leq 1\}$. (Observe that 0 has a basis of such neighborhoods). The semi-variation of a function $f \geq 0$ with respect to V is defined, when f is lower semi-continuous, by:

$$\mu^{\bullet}_V(f) = \sup_{|\varphi| \leq f} |\mu(\varphi)|_V$$

and for the other functions the definition is extended as in definition 1.1. We then have the following properties:

4.5 Theorem 1) a) $f \leq g$ implies $\mu^{\bullet}_V(f) \leq \mu^{\bullet}_V(g)$. When $(f_i)_{i \in I}$ is an increasingly directed family of ℓ.s.c. functions with upper-bound f,

$$\mu^{\bullet}_V(f) = \sup_{i \in I} \mu^{\bullet}_V(f_i)$$

b) $\mu^{\bullet}_V(\lambda f) = \lambda\, \mu^{\bullet}_V(f) \quad \lambda \geq 0 \quad (0x + \infty = 0)$

$\mu^{\bullet}_{\alpha V}(f) = \frac{1}{\alpha}\, \mu^{\bullet}_V(f) \quad \alpha > 0$

c) $W + W \subset V$ (W closed balanced) implies

$\mu^{\bullet}_V(f_1+f_2) \leq \sup(\mu^{\bullet}_W(f_1), \mu^{\bullet}_W(f_2))$

d) $\mu^{\bullet}_V(f) = 0 \;\forall\; V$ is equivalent to $f(t) = 0$ μ a.e.

e) $\mu^{\bullet}_V(f) < +\infty \;\forall\; V$ implies $f(t) < +\infty$ μ a.e.

2) Let $\mathcal{F}^{\bullet}(\mu)$ be the set of all complex functions such that $\mu^{\bullet}_{V}(|f|) < +\infty$ for all V, and let $\mathcal{F}^{\bullet}_{V} = \{f \in \mathcal{F}(\mu) : \mu^{\bullet}_{V}(|f|) < 1\}$. Then

a) $V_1 \subset V_2$ implies $\mathcal{F}^{\bullet}_{V_1} \subset \mathcal{F}^{\bullet}_{V_2}$

b) $\alpha \mathcal{F}^{\bullet}_{V} = \mathcal{F}^{\bullet}_{\alpha V}$

c) $W + W \subset V$ implies $\mathcal{F}^{\bullet}_{W} + \mathcal{F}^{\bullet}_{W} \subset \mathcal{F}^{\bullet}_{V}$

Thus $\mathcal{F}^{\bullet}(\mu)$ is a linear space (i.e. is stable under addition of functions and multiplication by scalars) and the sets $\mathcal{F}^{\bullet}_{V}$ form a basis of neighborhoods of zero for a topology compatible with the vector space structure, by virtue of which $\mathcal{F}^{\bullet}(\mu)$ will now be regarded as a t.v.s.

3) If $(| \ |_{d_i})_{i \in I}$ is a fundamental family of bounded continuous pseudo-metrics on E, the maps $f \longrightarrow \mu^{\bullet}_{d_i}(|f|)$ form a fundamental family of continuous pseudo-metrics in $\mathcal{F}^{\bullet}(\mu)$. In particular, when E is metrisablé the separated space associated with $\mathcal{F}^{\bullet}(\mu)$ is metrisable and $\mathcal{F}^{\bullet}(\mu)$ is complete.

4) $\mathcal{F}^{\bullet}(\mu)$ contains all bounded functions with compact support, in particular $\mathcal{K}(T) \subset \mathcal{F}^{\bullet}(\mu)$. The space $\mathcal{L}^1(\mu)$ is a closed linear topological subspace of $\mathcal{F}^{\bullet}(\mu)$, precisely the closure of $\mathcal{K}(T)$.

<u>Proof</u> The proof of 1) is similar to the proof of 1.2. It is convenient to leave the proof of 1) d) and e) until after 3) since for $f \in \mathcal{F}^{\bullet}(\mu)$ the statement d) is then obvious. For 3) observe that the maps $f \longrightarrow \mu^{\bullet}_{d}(|f|)$ are invariant pseudo-metrics on $\mathcal{F}^{\bullet}(\mu)$ (finite since by assumption $\sup_{x \in E} |x|_d < +\infty$). If $V = \{x : |x|_d \leq \varepsilon\}$, $\mu^{\bullet}_{V}(|f|) < 1$ implies $\mu^{\bullet}_{d}(|f|) \leq \varepsilon$. Thus they

are continuous pseudo-metrics. Conversely if V is given and $|x|_{d_i} < \varepsilon_i$, $i = 1,\dots,n$ implies $x \in V$, $\mu^{\bullet}_{d_i}(|f|) < \varepsilon_i$ implies $\mu^{\bullet}_V(|f|) \leq 1$. The last assertion 4) now follows immediately from this. We leave the details of the proof to the reader.

In terms of the space $\mathcal{F}^{\bullet}(\mu)$ the Lusin property can be formulated as follows: Let K be a compact set and let $(K_i)_{i\in I}$ be the increasingly directed family of compact subsets $K' \subset K$ such that $f_{|K'}$ is continuous. Then f possesses the Lusin property with respect to μ iff χ_K is the limit of the χ_{K_i} in the space $\mathcal{F}^{\bullet}(\mu)$. When μ is a Radon measure χ_K and χ_{K_i} belong to $\mathcal{L}^1(\mu)$ and the consideration of $\mathcal{F}^{\bullet}(\mu)$ is not necessary.

Observe that when E is a metric topological vector space with a p-norm i.e a continuous pseudo-metric such that $|\lambda x| = |\lambda|^p|x|$ for all $\lambda \in \mathbb{C}$ and $x \in E$, we have, if we put $V = \{x : |x| \leq 1\}$ $|x|_V = |x|^{1/p}$, whence $\mu^{\bullet}_V = (\mu^{\bullet})^{1/p}$. In that case $\mathcal{F}^{\bullet}(\mu)$ can be described directly in terms of the given p-norm as

$$\mathcal{F}^{\bullet}(\mu) = \{f : \mu^{\bullet}(|f|) < +\infty\}$$

and $\mathcal{L}^1(\mu)$ as the closure of $\mathcal{K}(T)$ in $\mathcal{F}^{\bullet}(\mu)$. In that case $\mu^{\bullet}(|\lambda f|) = |\lambda|^p \mu^{\bullet}(|f|)$ so that the semi-variation defines a p-norm in $L^1(\mu)$. When $p = 1$, i.e. when E is normed, $L^1(\mu)$ is a Banach space.

§5 Measures with values in C-spaces.

In this section we characterize the spaces E such that arbitrary E-valued Radon maps are Radon measures.

5.1 Definition Let E be an arbitrary t.v.s. A sequence $(x_n)_{n\in N}$ is said to be a C-sequence when the finite sums $\sum_n \lambda_n x_n$ remain bounded in E for $|\lambda_n| \leq 1$, (i.e when the discrete measure with mass x_n at n is bounded).

The space E is said to be a C-space when every C-sequence tends to zero.

Remark When E is quasi-complete (or sequentially complete) $(x_n)_{n\in N}$ is a C-sequence iff $(c_n x_n)_{n\in N}$ is summable for every sequence (c_n) tending to zero (see Theorem 4.4). In that case when E is a C-space every C-sequence is summable and even B-summable(1). Thus we are following in that case the definitions of L. Schwartz [11].

5.2 Theorem Let E be a C-space and let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon map. Then μ is a Radon measure. If μ is bounded μ has the global extension property.

Proof Let $\varphi_n \in \mathcal{K}(T)$ with $\Sigma|\varphi_n(t)| \leq \chi_K(t)$ (resp. $\Sigma|\varphi_n(t)| \leq 1$). Then $(\varphi_n)_n$ is a C-sequence in $\mathcal{K}(T, K)$ (resp. in $\mathcal{C}_o(T)$), consequently $(\mu(\varphi_n))_n$ is a C-sequence in E, whence $\lim_n \mu(\varphi_n) = 0$

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(1) Observe that if the finite sets K_n are disjoint the sequence $y_n = \sum_{i\in K_n} x_i$ is also a C-sequence. Thus if every C-sequence tends to zero, every C-sequence satisfies Cauchy's condition for summability, and is summable. Also, if (x_n) is a C-sequence so is $(\lambda_n x_n)_n$ with $(\lambda_n) \in \ell^\infty$. Thus $(\lambda_n x_n)$ is summable and (x_n) is B-summable (4.4 is not needed here).

and the conclusion follows from Theorem 4.3.

5.3 Theorem Let E be a quasi-complete C-space, and let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon map. Then a function f is μ-integrable iff it is μ-measurable and $\mu_V^\bullet(|f|) < +\infty$ for all V. In particular Borel functions such that $\mu_V^\bullet(|f|) < +\infty$ for all V are μ-integrable.

5.4 Corollary. Assume the topology of the C-space E is defined by a p-norm $(|x+y| \leq |x|+|y|,\ |\lambda x| = |\lambda|^p|x|)$. Then every μ-measurable function, in particular every Borel function, f such that $\mu^\bullet(|f|) < +\infty$ is μ-integrable. Furthermore if $(f_n)_{n\geq 1}$ is a sequence of nonnegative μ-integrable functions such that $\underline{\lim}_n\, \mu^\bullet(f_n) < +\infty$, $f(t) = \underline{\lim}_n\, f_n(t)$ is finite μ a.e, f is μ-integrable and $\mu^\bullet(f) \leq \underline{\lim}\, \mu^\bullet(f_n)$.

Proof The conditions of Theorem 5.3 are obviously necessary. To prove the sufficiency let $(K_n)_{n\geq 1}$ be a sequence of disjoint compact sets such that $f/_{K_n}$ is continuous, and let $f_n = \chi_{K_n} f$. Then $|\Sigma \lambda_n f_n| \leq |f|$ for any finite sequence (λ_n) with $|\lambda_n| \leq 1$, whence $\mu_V^\bullet(|\Sigma \lambda_n f_n|) \leq \mu_V^\bullet(|f|)$; thus $(f_n)_{n\geq 1}$ is a C-sequence in $\mathcal{L}^1(\mu)$, whence $(\int f_n d\mu)_{n\geq 1}$ is a C-sequence in E and consequently B-summable which implies that f is μ-integrable by Theorem 4.3 5).

The corollary now follows from the remarks in §4 and Theorem 2.16. Without using the semi-variations $\mu_V^\bullet$ it can be proved directly using the following lemma the proof of which is left to the reader:

Lemma If $|\lambda x| = |\lambda|^p|x|$, $\mu^\bullet(|\lambda f|) = |\lambda|^p\, \mu^\bullet(|f|)$, $\mu^\bullet(f) < +\infty$ implies $f(t) < \infty$ a.e. A set $H \subset \mathcal{L}^1(\mu)$ is bounded iff

$$\sup_{f \in H} \mu^{\bullet}(|f|) < +\infty .$$

Examples of C-spaces.

1) Locally convex C-spaces.

Proposition Let E be a quasi-complete (or even sequentially complete) locally convex space. A sequence $(x_n)_{n \geq 1}$ is a C-sequence iff $\sum_n |< x_n, x' >| < +\infty$ for every continuous linear form $x' \in E'$. E is a C-space iff for every C-sequence there exists $x \in E$ such that

$$< x, x' > = \sum_n < x_n, x' > \quad \text{for all} \quad x' \in E' .$$

In particular if E is weakly sequentially complete, and more particularly if E is semi-reflexive, E is a C-space(1).

Proof The finite sums $\Sigma_n \lambda_n x_n$ are bounded iff they are weakly bounded; but $\sup_{|\lambda_n| \leq 1} |< \Sigma \lambda_n x_n , x' >| = \sum_n |< x_n, x' >|$, whence the first assertion. If E is a C-space $x = \Sigma x_n$ exists in E, a fortiori $< x, x' > = \sum_n < x_n, x' >$. Conversely the hypothesis implies that for every subset $A \subset N$, there exists $S_A \in E$ such that $< S_A, x' > = \sum_{n \in A} < x_n, x' >$ whence (x_n) is summable by Orlicz theorem.

If E is a separable Banach space

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(1) In [14], spaces with this property are termed weakly Σ-complete.

which is a dual space E is also a C-space. This can be proved using a theorem of R. S. Phillips [9] p. 530, according to which every continuous linear map from $c_o(N)$ to E is compact, and Theorem 4.4, or by using a generalization of Orlicz' theorem (1).

In particular the spaces L^p, $1 \leq p < +\infty$ are C-spaces. (Whereas $\mathcal{C}(K)$ and L^{∞}, isomorphic to a space $\mathcal{C}(K)$, is not, unless K is finite, since the identity map $\mathcal{C}(K) \longrightarrow \mathcal{C}(K)$ does not have the extension property.

2) L. Schwartz [11] has shown that the spaces $L^p(\nu)$ $(0 < p < 1, \nu \geq 0)$ and $L^o(\nu)$, the space of equivalence classes of ν-measurable functions with the topology of convergence in measure (ν bounded) are C-spaces.

Clearly if the topology of E is the coarsest for which linear maps $u_i : E \dashrightarrow E_i$ into C-spaces are continuous, E is also a C-space. Thus if ν is unbounded the space $L^o(\nu)$ with the topology of convergence in measure on every set of finite measure (or on every compact set if ν is a positive Radon measure) is a C-space.

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(1) See [14] p. 141 and 180.

3) Orlicz spaces

Matuszewska and Orlicz [7] have characterised a class of modular spaces with the property (O) that all sequences $(x_n)_{n \in N}$ for which the finite sum $\sum_{n \in K} x_n$ are bounded are summable. A space with the property (O) is a fortiori a C-space. Matuszewska and Orlicz show that a space $L^{*\varphi}(\mu)$ has the property (O) if and only if there exists k such that $\varphi(2u) \leq k\,\varphi(u)$ for u sufficiently large, ([7] Theorem 4). Thus these spaces $L^{*\varphi}(\mu)$ are C-spaces.

Remark When E is a C-space and μ is an E valued Radon measure it has been shown that $L^1(\mu)$ is a C-space at least when E is a Banach space. Whether this is the case in general is an unsolved question even when E is metrisable.

By definition E is a C-space iff every bounded discrete Radon map $\mu : \mathcal{K}(N) \longrightarrow E$ has the global extension property. Thus the affirmations of 5.2 are true only for C-spaces. Applying the following theorem with $E = F$, u the identity map it is seen that at least among sequentially complete spaces[1] the C-spaces are the only spaces for which an arbitrary Radon map $\mu : \mathcal{C}[0, 1] \longrightarrow E$ is a Radon measure.

5.5 Theorem Let E be a sequentially complete t.v.s, let F be

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[1] The completion of a metrisable C-space is easily seen to be a C-space.

a t.v.s and $u : E \longrightarrow F$ a continuous linear map. Then the following conditions are equivalent:

1) For every E-valued Radon map μ, $u \circ \mu$ is a Radon measure.

2) For every bounded E-valued Radon map μ, $u \circ \mu$ has the global extension property.

3) For every Radon map $\mu : \mathcal{C}[0, 1] \longrightarrow E$, $u \circ \mu$ is a Radon measure.

4) For every C-sequence $(x_n)_{n \in N}$ in E, $\lim_{n \to \infty} u(x_n) = 0$.

Here $[0, 1]$ may be replaced by any other infinite metrisable compact space.

Proof 4) $\Longrightarrow$ 2) $\Longrightarrow$ 1) $\Longrightarrow$ 3 $\Longrightarrow$ 4).

4) $\Longrightarrow$ 2) Let $\varphi_n \in \mathcal{K}(T)$ $\Sigma|\varphi_n(t)| \leq 1$. Then (φ_n) is a C-sequence in $\mathcal{C}_o(T)$ hence $(\mu(\varphi_n))$ is a C-sequence in E and $(u \circ \mu)(\varphi_n)$ tends to zero.

2) $\Longrightarrow$ 1) If $\mu : \mathcal{K}(T) \longrightarrow E$ is a Radon measure the restriction to $\mathcal{K}(\omega)$ where ω is a relatively compact open set is a bounded Radon map, hence measure, and it follows that μ is itself a Radon measure.

3) $\Longrightarrow$ 4) Let K be a compact metrisable space and assume every Radon map $\mu : \mathcal{C}(K) \longrightarrow E$, $u \circ \mu$ is a Radon measure. Let $(x_n)_{n \geq 1}$ be a C-sequence in E. Let $(t_n)_{n \geq 1}$ be a convergent sequence in K with limit t and $t_n \neq t_m$ for $n \neq m$ and let $p : \mathcal{C}(K) \longrightarrow c_o$ be the map defined by $p(\varphi) = \{\varphi(t_n) - \varphi(t)\}_{n \geq 1}$. Let $\mu(\varphi) = \sum_n [\varphi(t_n) - \varphi(t)]x_n$. Then $\mu : \mathcal{C}(K) \longrightarrow E$ is a Radon map and by hypothesis $u \circ \mu$ is a Radon measure and clearly $u \circ \mu(\{t_n\}) = u(x_n)$. Consequently $\lim_n u(x_n) = 0$.

§6 The use of weaker topologies.

In the case of a locally convex space E the weak topology is useful in integration theory as is shown by previous work in the subject which usually relies on the existance of continuous linear forms. In the general case one may still make use of weaker topologies. Thus in L^p $0 < p \leq +\infty$ we may consider the topology of convergence in measure. When E is locally convex it can be useful to consider topologies which are weaker than $\sigma(E, E')$ but still separated.

In what follows E and F will be separated topological vector spaces with a continuous injective map $j : E \hookrightarrow F$ by means of which E is usually identified with a linear subspace of F. We assume always that E is quasi-complete. No such hypothesis is necessary on F; we can always complete F without modifying the assumption. Sometimes the space F coincides with E as a linear space but is equipped with a weaker topology.

Given a Radon map $\mu : \mathcal{K}(T) \longrightarrow E$ we write $\tilde{\mu} = j \circ \mu$. Then by Theorem 4.1 $\mathcal{L}^1(\mu) \subsetneq \mathcal{L}^1(\tilde{\mu})$ which gives rise to a natural map $L^1(\mu) \longrightarrow L^1(\tilde{\mu})$, and $\int f\, d\tilde{\mu} = j \int f\, d\mu$ for all $f \in \mathcal{L}^1(\mu)$. When identifying E with a subspace of F we write $\int f\, d\tilde{\mu} = \int f\, d\mu$. When μ is a Radon measure, so is $\tilde{\mu}$ and every μ-measurable (resp. μ-negligible) function is $\tilde{\mu}$-measurable (resp. $\tilde{\mu}$-negligible) by Theorem 4.1.

6.1 Theorem Let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon measure, E being a quasi-complete separated t.v.s. Assume $E \underset{j}{\hookrightarrow} F$ and let $\tilde{\mu} = j \circ \mu$. Then a function or set is μ-measurable (resp. μ-negligible) if and only if it is $\tilde{\mu}$-measurable (resp. $\tilde{\mu}$-negligible).

Proof Lemma 1 The natural map $L^1(\mu) \longrightarrow L^1(\tilde{\mu})$ is injective.

Let V be a neighborhood of zero in E and let
$\mathcal{U}_V = \{F \in L^1(\mu) : \int \varphi f \, d\mu \in V \ \forall f \in F \ \forall \varphi \in \mathcal{K}(T), |\varphi| \leq 1\}$.

Then by Theorem 1.4 d) the sets $\mathcal{U}_V$ constitute a fundamental set of neighborhoods of zero in $L^1(\mu)$ when V runs through such a fundamental set in E. Assume that the image of $F \in L^1(\mu)$ in $L^1(\tilde{\mu})$ is zero, i.e $f(t) = 0$ $\tilde{\mu}$ a.e for $f \in F$. Then $\int \varphi f \, d\tilde{\mu} = j \int \varphi f \, d\mu = 0$ for all φ, and j being injective $F \in \mathcal{U}_V$ for all V, whence $F = 0$.

Lemma 2 Let $(F_i)_{i \in I}$ be a bounded Cauchy net in $L^1(\mu)$ and assume $F_i \longrightarrow 0$ for the topology induced by $L^1(\tilde{\mu})$. Then $F_i \longrightarrow 0$ in $L^1(\mu)$.

Proof Let V be a closed neighborhood of zero in E. By hypothesis there exists i_o such that $F_i - F_j \in \mathcal{U}_V$ for $i, j \geq i_o$, that is, if $f_i \in F_i$, $\int \varphi f_i d\mu - \int \varphi f_j d\mu \in V$ for all $\varphi \in \mathcal{K}(T)$ with $|\varphi| \leq 1$. Now $(\int \varphi f_i d\mu)_{i \in I}$ is a bounded Cauchy net in E, hence converges to some limit $\nu(\varphi)$. But $\int \varphi f_i d\mu = \int \varphi f_i d\tilde{\mu}$ tends to zero in F. Consequently $\nu(\varphi) = 0$ and $\int \varphi f_i d\mu$ tends to zero in E, thus V being closed we have, letting j go to infinity, $\int \varphi f_i d\mu \in V$ for all $i \geq i_o$ and $|\varphi| \leq 1$, that is $F_i \in \mathcal{U}_V$ for all $i \geq i_o$, which proves the lemma.

Lemma 3 Let $F_i, F_o \in L^1(\mu)$ and assume $(F_i)_{i \in I}$ is an increasingly directed family such that $0 \leq F_i \leq F_o$. Then $(F_i)_{i \in I}$ is a bounded Cauchy net.

Clearly F_i is bounded, for the topology as well as for the order; let $f_i \in F_i$, $f \in F_o$ and assume (f_i) is not a Cauchy net. Then there exists an invariant metric d, and a number $\alpha > 0$ and a sequence $(f_{i_k})_{k \in N}$ such that $\mu_d^{\cdot}(|f_{i_{k+1}} - f_{i_k}|) > \alpha$, and $f_{i_k}(t) \leq f_{i_{k+1}}(t) \leq f(t)$ μ.a.e. But then $(f_{i_k})_k$ converges by the dominated convergence theorem which is a contradiction.

Now let f be a $\tilde{\mu}$-measurable function, let K be a compact set and let $(K_i)_{i \in I}$ be the family of compact subsets K' of K such that $f_{|K'}$ is continuous. Since the union of two sets of this family belongs to the same family it is an increasingly directed. Let F_i (resp. F_o) be the class of χ_{K_i} (resp. χ_K) in $L^1(\mu)$. By lemma 3 $(F_i)_{i \in I}$ is a Cauchy net in $L^1(\mu)$, but since f is $\tilde{\mu}$-measurable F_i tends to F_o and $F_o - F_i$ tends to zero, for the topology induced by $L^1(\tilde{\mu})$. Thus by Lemma 2 F_i tends to F_o in $L^1(\mu)$, which proves that f is μ-measurable. Let A be a $\tilde{\mu}$-negligible set, then A is $\tilde{\mu}$-measurable, hence μ-measurable and for any compact K, $\chi_{A \cap K} \in \mathcal{L}^1(\mu)$. By Lemma 1 $A \cap K$ is μ-negligible, and K being arbitrary, A is μ-negligible. This ends the proof.

6.2 Corollary. Let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon measure with values in a quasi-complete t.v.s. Let $(u_i)_{i \in I}$ be a family of continuous linear maps $u_i : E \longrightarrow E_i$ which separates the points of E. Put $\mu_i = u_i \circ \mu$. Then a function or set is μ-measurable (resp. μ-negligible) iff it is μ_i-measurable (resp. μ_i- negligible) for all i.

Proof Take $F = \prod_{i \in I} E_i$ and $j(x) = (u_i(x))_{i \in I}$. Then by Theorem 4.1 the hypothesis expresses that the function is $\tilde{\mu}$-measurable (resp. $\tilde{\mu}$-negligible).

We now give an example to show that the conclusion of theorem 6.1 does not hold in the absence of some completeness hypothesis on E, even if we assume $\int f\,d\mu \in E$ for all $f \in \mathcal{L}^1(\mu)$. Let E be the space of all universally measurable and bounded functions on $[0, 1]$ with the topology determined by the semi-norms $f \longrightarrow \int |f|\,d\nu$ where ν is an arbitrary positive measure on $[0, 1]$. Let $\mu : \mathcal{C}[0,1] \longrightarrow E$ be the canonical injection. Then by 4.1 $\mathcal{L}^1(\mu) = \bigcap_{\nu} \mathcal{L}^1(\nu) = E$ and the map $f \longrightarrow \int f\,d\mu$ is the identity map in E. Let F be the space of all functions with the topology of pointwise convergence. Then by 4.1 $\mathcal{L}^1(\tilde{\mu}) = \bigcap_{\nu \in D} \mathcal{L}^1(\nu) = F$ where D is the set of discreet measures. Hence an arbitrary function is $\tilde{\mu}$-measurable but only universally measurable functions are μ-measurable. This is also an example of a space $L^1(\mu)$ which is not quasi-complete.

Although in general $\tilde{\mu}$-integrable functions are not μ-integrable, in many cases a $\tilde{\mu}$-integrable function f will be μ-integrable provided the 'weak' integrals $\int gf\,d\tilde{\mu}$ belong to E. When this is true for discreet measures the following condition is satisfied:

(H) Every sequence $(x_n)_{n \geq 1}$ which is B-summable in F and such that $\Sigma_n \lambda_n x_n \in E$ for all $(\lambda_n) \in \ell^\infty$, is B-summable in E.

Conversely we have:

6.3 Theorem Let E be a quasi-complete t.v.s with $E \underset{j}{\hookrightarrow} F$, let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon measure and $\tilde{\mu} = j \circ \mu$. Assume that (E, F) satisfies the condition (H). Then $f \in \mathcal{L}^1(\mu)$ iff $f \in \mathcal{L}^1(\tilde{\mu})$ and $\int gf\,d\tilde{\mu} \in E$ for all bounded Borel functions g.

Proof If $f \in \mathcal{L}^1(\mu)$ $\int gf\,d\tilde{\mu} = \int gf\,d\mu \in E$. Conversely if $f \in \mathcal{L}^1(\tilde{\mu})$, f is μ-measurable by Theorem 6.1, and if $(K_n)_{n \geq 1}$ is a sequence of disjoint compact sets such that $f/_{K_n}$ is continuous the sequence $(\int_{K_n} f\, d\mu)_{n\geq 1} = (\int_{K_n} f\, d\tilde{\mu})_{n\geq 1}$ is B-summable in F and the sums $\Sigma \lambda_n \int_{K_n} f\, d\tilde{\mu} = \int gf d\tilde{\mu}$, where $g = \Sigma \lambda_n \chi_{K_n}$, belong to E. Thus (H) implies that $(\int_{K_n} f\, d\mu)_{n\geq 1}$ is B-summable in E, whence $f \in \mathcal{L}^1(\mu)$ by Theorem 4.3 5).

In order to describe conditions which are sufficient for (H) to hold we introduce the following terminology: We shall say that the pair (E, F) possesses the <u>automatic continuity</u> property if:

(AC) Every linear map u from a Banach space to E such that $j \circ u$ is continuous, is continuous. This holds in the following cases:

1) E is a complete metrisable t.v.s.

2) E is an $(\mathcal{L}\mathcal{F})$ - space.

3) E is a locally convex Suslin space.

4) E is locally convex and has a fundamental system of neighborhoods of zero closed for the topology induced by F.

5) Every subset of E which is bounded in F is bounded in E.

6) There exists on E a stronger t.v.s. topology satisfying (AC).

<u>Proof</u> Let $u : B \longrightarrow E$ be a linear map from a Banach space B to E such that $j \circ u$ is continuous. Then the graph of u is closed and 1) 2) 3) imply (AC) by the closed graph theorem [1]. Let V be an absolutely convex neighborhood of zero in E which is

(1) More generally (AC) holds whenever the closed graph theorem holds for B and E. This is the case here:
For 1) see Dunford and Schwartz [4] II 2.4.
For 2) see Grothendieck [6] chapter IV Theorem 2.
For 3) see L. Schwartz [13].

closed for the topology induced by F, whence $V = E \cap \overline{V}$ where $\overline{V}$ is closed in F. Then $u^{-1}(V) = (j \circ u)^{-1}(\overline{V})$ is absolutely convex, absorbing and closed in B, i.e a barrel and B being a Baire space it is a neighborhood of zero. 5) implies (AC) since u is continuous if u is bounded. Finally if u is continuous with respect to a stronger topology on E it is certainly continuous with respect to the given topology.

It may be said that all pairs (E, F) that arise in practice satisfy the condition (AC) [1].

We also use the following terminology: E is said to be <u>essentially separable</u> in F when every countable subset of E is contained in a separable subspace which is closed in E for the topology induced by F.

6.4 <u>Theorem</u> Assume (E, F) satisfies condition (AC) and that E is a C-space or that E is locally convex and essentially separable in F. Then (E, F) satisfies the condition (H).

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(1) If u is discontinuous there exists at least a continuous linear form x' on E (when E is locally convex) such that $x' \circ u$ is discontinuous. Such discontinuous linear forms do not arise naturally in analysis. Here is an example where (AC) is not satisfied: Let F be a Banach space and let E be the space F with the strongest locally convex topology. Then E is complete [6] ch. 4 §1 but the identity $F \longrightarrow E$ is discontinuous.

<u>Proof</u> Let (x_n) be B-summable in F and assume $S_\lambda = \sum_n \lambda_n x_n \in E$. Then by (AC) the map $\lambda \longrightarrow S_\lambda$ from ℓ^∞ to E is continuous, in particular (x_n) is a C-sequence in E. Thus if E is a C-space (x_n) is B-summable in E. If on the other hand E is locally convex and essentially separable in F, let E_1 be a separable subspace of E closed for the topology induced by F, containing each x_n and consequently $S_\lambda = \Sigma \lambda_n x_n$. Then the map $\lambda \longrightarrow S_\lambda$ from ℓ^∞ to E_1 is weakly compact [(1)], a fortiori its restriction to c_o is weakly compact whence (x_n) is B-summable (see 3.3 and 4.4). This proves the theorem.

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(1) See Grothendieck [5] p. 168 Corollary 1.

Thus in particular we have the following:

6.5 Theorem Let E be a quasi-complete t.v.s. with $E \underset{j}{\hookrightarrow} F$. Let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon measure and $\tilde{\mu} = j \circ \mu$. Assume E satisfies one of the following conditions

a) E is a metrisable C-space.

b) E is a locally convex Suslin space [(1)].

c) E is a separable or weakly sequentially complete $\mathcal{LF}$ - space.

Then a function f is μ-integrable iff it is $\tilde{\mu}$-integrable and $\int g f d\tilde{\mu} \in E$ for each bounded Borel function. In particular, assume b) or c) is satisfied and let $(x_i')_{i \in I}$ be a family of continuous linear forms on E separating the points of E. Put $\mu_i = x_i' \circ \mu$. Then $f \in \mathcal{L}^1(\mu)$ iff $f \in \mathcal{L}^1(\mu_i)$ for each i and for every bounded Borel function g there exists $\nu(g) \in E$ such that $< \nu(g), x_i' > = \int g f \, d\mu_i$ for all i.

The last assertion is an immediate consequence of the preceding one by taking $F = \mathbb{C}^I$ $j(x) = (< x, x_i >)_{i \in I}$

Loosely speaking this means that f is μ-integrable as soon as one can make sense of the proposed integrals $\int g f d\mu$ as elements of E.

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(1) For examples of Suslin spaces arising in analysis see L. Schwartz [13].

A typical space to which the above theorem does not apply is the space L^{∞} (for instance if $e_n = (\delta_{n,i})_i$ in ℓ^{∞}, (e_n) is B-summable for the topology $\sigma(\ell^{\infty}, \ell^{1})$ but not in the norm since $\| e_n \| = 1$). However if $E = G'$ is the dual of a Banach space and is equipped with the weak $*$ topology, or even $\tau(E, G)$, E is a C-space and for any F such that $E \hookrightarrow F$ the pair satifies AC (by the 6^{th} condition insuring AC), hence (E, F) satisfies (H).

In several cases the conditions $\int g f d\tilde{\mu} \in E$ can be relaxed somewhat: When E is a C-space and (AC) holds it suffices to have $\int \varphi f d\tilde{\mu} \in E$ for $\varphi \in \mathcal{C}_o(T)$ (since the Radon map $\varphi \dashrightarrow \int \varphi f d\tilde{\mu} \in E$ then has the global extension property). In many cases it suffices to have $\int_A f d\tilde{\mu} \in E$ for all Borel sets. This occurs when E is locally pseudo-convex and the topology induced on E by F possesses the Orlicz property [(1)], and notably in the following cases: E and F are locally convex, E is a Frechet space and essentially separable in F or weakly sequentially complete. The topology induced on $\ell^{p}(I)$ by $\mathbb{C}^{I}$ $(0 < p < +\infty)$ possesses the Orlicz property [(2)]. It has been proved by Ph. Turpin [(3)] that the topology induced by $L^{o}(\nu)$ on $L^{p}(\nu)$ $0 < p < +\infty$ possesses the Orlicz property. When E and F are locally convex and E possesses a fundamental system of neighborhoods closed for the topology induced by F, and is essentially separable in F or weakly Σ-complete, it suffices to

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(1) See [14] Appendix II and [16] § 0

(2) See [15] Theorem 4.

(3) Oral communication.

have $\int_\omega f \, d\tilde{\mu} \in E$ for every open set ω (1).

As an example of a not necessarily locally convex case we mention the following:

<u>Example</u> Let $\mu = (\mu_i)_{i \in I}$ be a Radon map with values in $\ell^p(I)$ $(0 < p < +\infty)$, i.e. a family of complex measures such that $\sum_{i \in I} |\mu_i(\varphi)|^p < +\infty$ for all $\varphi \in \mathcal{K}(T)$. Then $f \in \mathcal{L}^1(\mu)$ iff $f \in \mathcal{L}^1(\mu_i)$ for all i and $\sum_{i \in I} |\int_A f \, d\mu_i|^p < +\infty$ for all Borel sets A, or $\sum_{i \in I} |\int \varphi f d\mu_i|^p < +\infty$ for all $\varphi \in \mathcal{C}_o(T)$.

Next we describe cases where every $\tilde{\mu}$-integrable function is actually μ-integrable.

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(1) See [14] 3.20 and II.3.

6.6 Theorem Let E be a quasi-complete t.v.s with $E \underset{j}{\hookrightarrow} F$. Assume a) that E is a C-space b) that every subset of E which is bounded in F is bounded in E. Then every $\tilde{\mu}$-integrable function is μ-integrable. (i.e. $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\tilde{\mu})$ coincide as linear spaces, but not in general as topological spaces).

Proof Let $f \in \mathcal{L}^1(\tilde{\mu})$. Then f is μ-measurable by 6.1. Let $(K_n)_n$ be a sequence of disjoint compact sets such that $f/_{K_n}$ is continuous. Then the sequence $\left(\int_{K_n} f\, d\mu\right)_n = \left(\int_{K_n} f\, d\tilde{\mu}\right)_n$ is a C-sequence in F, thus the finite sums $\Sigma \lambda_n \int_{K_n} f\, d\mu$ with $|\lambda_n| \leq 1$ are bounded in F, hence in E. Consequently $\left(\int_{K_n} f\, d\mu\right)_n$ is a C-sequence in E and therefore B-summable whence $f \in \mathcal{L}^1(\mu)$ by Theorem 4.3 5).

6.7 Corollary Let E be a locally convex quasi-complete C-space, and let μ be an E-valued Radon measure. Then f is μ-integrable iff f is $x' \circ \mu$ - integrable for all $x' \in E'$. If E is a Banach space it suffices to have $f \in \bigcap_{x' \in H} \mathcal{L}^1(x' \circ \mu)$ where H is a closed norm determining subspace of E'. (1)

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(1) This was already proved in [14]. Recall that a locally convex space is a C-space if it is weakly Σ-complete, in particular when it is weakly sequentially complete.

<u>Remark</u> Let (Ω, P) be a probability space and consider the Radon measure $\mu : \mathcal{K}(\mathbb{R}) \longrightarrow L^2(\Omega, P)$ associated with Brownian motion $x(t, \omega)$ by $\mu(\varphi) = \int \varphi(t)\,dx(t)$ (see example 3.8). Let $\tilde{\mu}$ be the corresponding measure with values in $L^0(\Omega, P)$. Then we still have $\mathcal{L}^1(\tilde{\mu}) = \mathcal{L}^1(\mu)$ even as a t.v.s. Indeed if $f \in \mathcal{L}^1(\tilde{\mu})$ f is μ-measurable, and if $(K_n)_n$ is a sequence of disjoint compact subsets such that $f/_{K_n}$ is continuous the sequence $\left(\int_{K_n} f\,d\mu\right) = \left(\int_{K_n} f\,d\tilde{\mu}\right)$ is a sequence of mutually orthogonal and therefore independent Gaussian random variables, summable in $L^0(\Omega)$ and therefore in L^2, whence $\lim_n \int_{K_n} f\,d\mu = 0$ in L^2 and $f \in \mathcal{L}^1(\mu)$. Thus the map $L^1(\mu) \longrightarrow L^1(\tilde{\mu})$ is bijective and since both spaces are complete and metrisable it is an isomorphism. Hence $L^1(\mu) = L^1(\tilde{\mu}) = L^2(\mathbb{R})$.

Up to now it has been assumed that μ is a Radon measure (in the preceding example ℓ^p is a C-space). However when (E, F) satisfies condition (H) one may verify this as follows:

6.8 <u>Theorem</u> Let E be a quasi-complete t.v.s with $E \hookrightarrow F$ and assume (E, F) satisfies condition (H). Let $\mu : \mathcal{K}(T) \longrightarrow E$ be a Radon map and $\tilde{\mu} = j \circ \mu$. Then μ is a Radon measure iff $\tilde{\mu}$ is a Radon measure and $\int f\,d\tilde{\mu} \in E$ for every bounded lower semi-continuous Baire function f with compact support.

This follows immediately from the next theorem applied to the restrictions of μ to $\mathcal{K}(\omega)$ where ω is a relatively compact open set.

6.9 <u>Theorem</u> Let E be quasi-complete with $E \hookrightarrow F$ and assume (E, F) satisfies condition (H). Let $\mu : \mathcal{K}(T) \longrightarrow E$ be a bounded Radon map and $\tilde{\mu} = j \circ u$. Then the following are equivalent:

1) $\mathcal{L}^1(\mu)$ contains all bounded Borel functions.

2) For every bounded lower semi-continuous Baire (1) function f, $f \in \mathcal{L}^1(\tilde{\mu})$ and $\int f \, d\tilde{\mu} \in E$.

3) For every $\varphi_n \in \mathcal{K}(T)$ with $0 \leq \varphi_n \leq \varphi_{n+1} \leq 1$ $\mu(\varphi_n)$ converges in F and $\lim_n \mu(\varphi_n) \in E$.

<u>Proof</u> 1) $\Longrightarrow$ 3) $\Longrightarrow$ 2) $\Longrightarrow$ 1), and it suffices to prove the last two implications.

Observe first that 3) implies that $\tilde{\mu}$ has the global extension property: it suffices to verify condition 9) in Theorem 3.3, with $\varphi_n \geq 0$, $\sum_{n=1}^{\infty} \varphi_n(t) \leq 1$. Let $\psi_n = \sum_{i=1}^{n} \varphi_i$, then by hypothesis $\tilde{\mu}(\psi_n)$ converges in F i.e $\sum_{i=1}^{\infty} \mu(\varphi_i)$ converges, a fortiori $\tilde{\mu}(\varphi_i)$ tends to zero. Thus if f is a lower semi-continuous bounded Baire function $f \in \mathcal{L}^1(\tilde{\mu})$. We may assume

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(1) We take the Baire sets to constitute the smallest σ - algebra containing the compact G_δ .

$f \geq 0$. Then there exists a sequence $(\varphi_n)_{n\geq 1}$ $0 \leq \varphi_n \leq \varphi_{n+1} \leq f$, $f = \sup \varphi_n$, and $\int f \, d\tilde{\mu} = \lim_n \mu(\varphi_n)$. Now by hypothesis this limit belongs to E. Thus 3) implies 2). To show that 2) implies 1) it suffices to prove that for $\varphi_n \geq 0$ $\Sigma \varphi_n(t) \leq 1$ $\mu(\varphi_n)$ tends to zero in E. Observe that 2) implies that $\tilde{\mu}$ is a Radon measure (Theorem 3.1) and that $1 \in \mathcal{L}^1(\tilde{\mu})$. Let $\lambda_n \geq 0$ $(\lambda_n) \in \ell^\infty$. Then $\Sigma \lambda_n \mu(\varphi_n) = \Sigma \lambda_n \tilde{\mu}(\varphi_n)$ converges in F by the dominated convergence theorem, and the sum is $\int f \, d\tilde{\mu}$ where $f(t) = \sum_n \lambda_n \varphi_n(t)$. Thus $\Sigma \lambda_n \mu(\varphi_n) \in E$ and by (H) $(\mu(\varphi_n))$ is B-summable in E, a fortiori $\lim_n \mu(\varphi_n) = 0$ in E. This ends the proof.

<u>Remark</u> When E is a quasi-complete locally convex space and F is E with the weak topology condition (H) is satisfied and the above condition 3) is one of the weak compactness criteria due to A. Grothendieck [(1)].

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(1) See A. Grothendieck [5] Theorem 6 p. 160.

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