Math 263, Section 4, Fall 2019

Course syllabus can be found on Blackboard under course content. The followings are some important facts:

Three Tests (no make up test), each counts 100 points, total give 300 points

One Final: counts 200 points (the percentage can be used to replace the lowest score of 3 tests)

12 homework sets counting a total of 100 points will be given (2 set dropped). Please go to pearsonmylabandmastering.com to do your homework.

The course ID number is wei93641.

Total points you can get are 600 points.

We will use grades A, A−, B+, B, B−, C+, C, D, F. See syllabus for more details.

Very important

1. Attending to the class and doing your homework. I will put my lecture notes and some other materials on Blackboard under the course content and make some announcements in the class.

2. I reserve the right to deduct some points from the total points of some one who has more than three absences.

3. Prepare 4 blue books to take the tests and final exam.

4. When solve a problem in a quiz or in a test, please solve it step by step. You can get some partial credits.

5. There will be no calculators or cellphones used during any test, exam, or in class assignment under ANY circumstances.

6. Cell phones, pagers, and other electronic devices that might cause disruption should be turned off or silenced before class begins.
7. Mathematics tutoring (FREE!) will occur in the J.D. Williams Library Commons. The Commons is on the bottom floor of the J.D. Williams Library. No appointment is necessary. A desk worker is stationed near the reference desk and can point you in the direction of a tutor.

Mondays to Thursdays from 10am to 7pm and Fridays from 10am to 2pm

8 Disability Access and Inclusion: The University of Mississippi is committed to the creation of inclusive learning environments for all students. If there are aspects of the instruction or design of this course that result in barriers to your full inclusion and participation, or to accurate assessment of your achievement, please contact the course instructor as soon as possible. Barriers may include, but are not necessarily limited to, timed exams and in-class assignments, difficulty with the acquisition of lecture content, inaccessible web content, and the use of non-captioned or non-transcribed video and audio files. If you are approved through SDS, you must log in to your Rebel Access portal at https://sds.olemiss.edu to request approved accommodations. If you are NOT approved through SDS, you must contact Student Disability Services at 662-915-7128 so the office can:

a. determine your eligibility for accommodations,
b. disseminate to your instructors a Faculty Notification Letter,
c. facilitate the removal of barriers, and
d. ensure you have equal access to the same opportunities for success that are available to all students.

Withdrawal deadline Date: Oct 7, 2019
The final will be at 4:00pm on Dec 13, 2019
Chapter 8 Sequences and Infinite Series

8.1 An Overview

Sequence: \( \{a_1, a_2, \cdots, a_n, \cdots\} = \{a_n\}_{n=1}^{\infty} \), where \( a_n \) is a real number, which is the \textbf{nth term}, for any integer \( n \), which is called an \textbf{index}.

\textbf{Explicit Formula}: \( a_n = f(n) \) for \( n = 1, 2, 3, \cdots \), (like a function).

\textbf{Recurrence Relation}: \( a_{n+1} = f(a_n) \) for \( n = 1, 2, \cdots \), where \( a_1 \) is given.

\textbf{Example 1}. Use the explicit formula to for \( \{a_n\}_{n=1}^{\infty} \) to write the first four terms of each sequence:

(a) \( a_n = \frac{1}{3^n - 1} \); (b) \( a_n = \cos(n\pi) \).

\textbf{Example 2}. Use the recurrence relation for \( \{a_n\}_{n=1}^{\infty} \) to write the first four terms of the sequences:

(a) \( a_{n+1} = 2a_n + 1; \ a_1 = 1 \); (b) \( a_{n+1} = 2a_n + 1; \ a_1 = -1 \).
Example 3. Consider the following sequences:
(a) \( \{1, 4, 7, 10, \cdots \} \);  
(b) \( \{-2, 5, 12, 19, \cdots \} \).

(i) Find the next two terms of the sequence.

(ii) Find the recurrence relation of the sequence.

(iii) Find the explicit formula for the nth term of the sequence.

If \( \lim_{n \to \infty} a_n = L \) \((L \text{ constant})\), \( \{a_n\}_{n=1}^\infty \) converges.
Otherwise \( \{a_n\}_{n=1}^\infty \) diverges.

Example 4. Write out the terms of the following sequences and determine which of them are convergent:
(a) \( \{\frac{n}{n+1}\}_{n=1}^\infty \);

(b) \( \{\frac{n}{2^n}\}_{n=0}^\infty \);

(c) \( \{a_n\}_{n=1}^\infty \), where \( a_{n+1} = -2a_n, \ a_1 = 1 \).
Infinite Series and the Sequence of Partial Sums

If we try to add the terms of an infinite sequence \( \{a_n\}_{n=1}^{\infty} \), we get an expression of the form \( a_1 + a_2 + \cdots + a_n + \cdots \). This is called an infinite series (or just a series) and denote

\[
\sum_{n=1}^{\infty} a_n.
\]

Set \( S_n = a_1 + a_2 + \cdots + a_n \), which is called the partial sum of the above series. \( \{S_n\}_{n=1}^{\infty} \) is a sequence and if \( \lim_{n \to \infty} S_n = L \) exists for a fixed number \( L \), then \( \sum_{n=1}^{\infty} a_n \) is convergent (otherwise it is divergent) and

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = L.
\]

**Example 1.** Consider the infinite series

\[
0.9 + 0.09 + 0.009 + 0.0009 + \cdots + \cdots
\]

where each term of the sum is \( 1/10 \) of the previous term.

a. Find \( S_n \) for \( n = 1, 2, 3, 4, 5 \).

b. Find \( \lim_{n \to \infty} S_n \).
Example 2. Consider the infinite series

\[ \sum_{k=1}^{\infty} \frac{1}{k(k+1)}. \]

a. Find \( S_n \) for \( n = 1, 2, 3, 4 \).

b. Find \( \lim_{n \to \infty} S_n \).

Summary

1. A sequence \( \{a_1, a_2, \ldots, a_n, \ldots\} \) is an ordered list of numbers.
2. An infinite series \( \sum_{k=1}^{\infty} a_k \) is a sum of numbers.
3. The sequence of partial sum \( S_n = a_1 + a_2 \cdots + a_n \) is used to evaluate the series \( \sum_{k=1}^{\infty} a_k \).
Section 8.2 Sequences

If \( \lim_{n \to \infty} a_n = L \) (\( L \) constant), \( \{a_n\}_{n=1}^{\infty} \) converges.

Otherwise \( \{a_n\}_{n=1}^{\infty} \) diverges.

Use functions to determine the convergence

**Theorem**

Suppose \( f(x) \) is a function such that \( a_n = f(n) \) for \( n \geq n_0 \), then \( f(x) \to L \) implies \( a_n \to L \).

**Example 1** Determine the convergence for each of the following sequences

(a) \( \{a_n = \frac{1}{n^r}\}_{n=1}^{\infty}, \ r \) is an integer;

(b) \( \{b_n = \frac{n^{10}}{e^n}\}_{n=1}^{\infty} \);

(c) \( \{c_n = \sin \frac{\pi}{n}\}_{n=3}^{\infty} \).
Properties.
If \( \{a_n\} \) and \( \{b_n\} \) are both convergent and \( c \) is a constant, then
\[
\lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n;
\]
\[
\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n;
\]
\[
\lim_{n \to \infty} (ca_n) = c \lim_{n \to \infty} a_n;
\]
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}, \text{ if } \lim_{n \to \infty} b_n \neq 0;
\]
\[
\lim_{n \to \infty} (a_n^p) = (\lim_{n \to \infty} a_n)^p \text{ if } p > 0 \text{ and } a_n > 0.
\]

Increasing or decreasing sequence
If \( \frac{a_{n+1}}{a_n} \leq 1 \) or \( \frac{a_{n+1}}{a_n} \geq 1 \) for all \( n \), then \( \{a_n\}_{n=1}^\infty \) is decreasing\(\text{ (nonincreasing)} \) or increasing\(\text{ (nondcreasing)} \). A sequence is either nonincreasing or nondecreasing is said to be monotonic.

Bounded sequence
If \( a_n \leq M \) for a constant \( M \) for any \( n \), then \( \{a_n\}_{n=1}^\infty \) is bounded above. If \( a_n \geq m \) for a constant \( m \) for any \( n \), then \( \{a_n\}_{n=1}^\infty \) is bounded below. If \( a_n \) is bounded both above and below, then \( \{a_n\} \) is a bounded sequence.

Theorem: Every bounded monotonic sequence converges.

Example 2. Determine the convergence of the the following sequences:

(a) \( \{a_n = \frac{2^n}{n!}\}_{n=1}^\infty \).

(b) \( a_1 = 2, a_{n+1} = \frac{a_n + 6}{2}, n = 2, 3, \ldots \)
Geometric Sequences

Theorem for geometric sequences:

Let \( r \) be a real number. Then

\[
\lim_{n \to \infty} r^n = \begin{cases} 
0, & \text{if } |r| < 1 \\ 
1, & \text{if } r = 1 \\ 
\text{DNE}, & \text{if } r \leq -1 \text{ or } r > 1.
\end{cases}
\]

If \( r > 0 \), then \( \{r^n\} \) converges or diverges monotonically. If \( r < 0 \), then \( \{r^n\} \) converges or diverges by oscillation.

**Example 3.** Graph the following geometric sequences and discuss their behavior:

\( a. \ \{0.75^n\}_{n=1}^{\infty} \); \( b. \ \{(-0.75)^n\}_{n=1}^{\infty} \);

\( c. \ \{1.15^n\}_{n=1}^{\infty} \); \( d. \ \{(-1.15)^n\}_{n=1}^{\infty} \).
**Squeeze Theorem:** \( a_n \leq b_n \leq c_n \) for \( n \geq N \), \( a_n \to L, c_n \to L \), then \( b_n \to L \).

If \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

**Example 4.** Determine the convergence for each of the following sequences

(a) \( \{ a_n = \frac{(-1)^n}{n} \}_{n=1}^\infty \); 

(b) \( \{ b_n = \frac{\sin n}{n^2 + 2} \}_{n=5}^\infty \);
8.3 Infinite Series

If we try to add the terms of an infinite sequence \( \{a_n\}_{n=1}^{\infty} \), we get an expression of the form \( a_1 + a_2 + \cdots + a_n + \cdots \). This is called an infinite series (or just a series) and denote

\[
\sum_{n=1}^{\infty} a_n.
\]

Set \( S_n = a_1 + a_2 + \cdots + a_n \), which is called the partial sum of the above series. \( \{S_n\}_{n=1}^{\infty} \) is a sequence and if \( \lim_{n \to \infty} S_n = L \) exists for a fixed number \( L \), then \( \sum_{n=1}^{\infty} a_n \) is convergent (otherwise it is divergent) and

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = L.
\]

The geometric series

A series in the form \( \sum_{n=0}^{\infty} ar^n \) for a constant \( a \neq 0 \) is an infinite series is a geometric series. (Note that \( n \) start from 0)

**Theorem 1.** A geometric series converges to \( \frac{a}{1-r} \) if \( |r| < 1 \) and diverges if \( |r| \geq 1 \).

**Example 1.** Investigate the convergence or divergence of the following series:

(a) \( \sum_{n=2}^{\infty} 5 \cdot \frac{1}{3^n} \);

(b) \( \sum_{n=0}^{\infty} 6 \left( -\frac{7}{3} \right)^n \);
Telescoping Series: The inner terms of the series cancel (or telescope) leaving a simple express for $S_n$.

Example 2. Evaluate the following series:

a. $\sum_{k=1}^{\infty} \left( \frac{1}{3^k} - \frac{1}{3^{k+1}} \right)$;

b. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$
8.4 The Divergence and Integral Tests

The Harmonic Series

The Series \( \sum_{k=1}^{\infty} \frac{1}{k} \) is called a harmonic series.

Properties

If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge, then \( \sum_{n=1}^{\infty} c a_n \) and \( \sum_{n=1}^{\infty} (a_n \pm b_n) \) converge for a constant \( c \) and

\[
\sum_{n=1}^{\infty} c a_n = c \left( \sum_{n=1}^{\infty} a_n \right);
\]

\[
\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n.
\]

Remark:

1. If \( \sum_{n=N+1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges;
2. If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both diverge, \( \sum_{n=1}^{\infty} (a_n \pm b_n) \) may converge.

Example 1. Find the sum of the series

\[
\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}
\]
For a given series $\sum_{n=1}^{\infty} a_n$, usually it is not easy to determine whether it is convergent. We need more tools.

**Theorem (The Divergence Test).** If a series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k \to \infty} a_k = 0$. That is if $\lim_{k \to \infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

**Remark:** If $\lim_{k \to \infty} a_k = 0$, $\sum_{k=1}^{\infty} a_k$ may be divergent.

**Theorem (The Integral Test).** Let $\{a_k\}_{k=1}^{\infty}$ be a sequence and $f(x)$ is a continuous positive decreasing function for $x \geq N$ such that $a_k = f(k)$. Then $\sum_{k=N}^{\infty} a_k$ and $\int_{N}^{\infty} f(x)dx$ both converge or diverge.

**Example 2.** Determine whether the following series converge or diverge:

(a) $\sum_{k=1}^{\infty} \frac{k^2}{k^2+3}$;

(b) $\sum_{k=1}^{\infty} \frac{1}{k^p}$;

(c) $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$. 
The $p$-series: $\sum_{k=1}^{\infty} \frac{1}{k^p}$.

**Theorem (Convergence of the $p$-series).**
The $p$-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergence if $p > 1$ and divergence if $p \leq 1$.

**The Estimating the Value of Infinite Series**

Suppose we have been able to use the Integral Test to show that a series is convergent. Now, we want to find an approximation to the sum $s$ of the series. In order to do this, we need to estimate the size of the remainder. As $a_n \geq 0$, we have

$$0 \leq R_n = s - S_n = \sum_{k=n+1}^{\infty} a_k \leq \int_{n}^{\infty} f(x)dx.$$

**Example 3.** How many terms are required to ensure the sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is accurate to within $10^{-5}$? %vskip 4.5cm
8. 5 The Ratio, Root and Comparison Test

Another useful test to determine whether a given series converges is the comparison test. The idea is to change a complicated looked series to a relatively simple and familiar looked series, which we can use to determine the convergence of the complicated one.

The ratio test

Given a series \( \sum_{n=1}^{\infty} a_n \) with \( a_n > 0 \) for any \( n \). Suppose that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L \). Then

(i) the series converges if \( L < 1 \);
(ii) the series diverges if \( L > 1 \) or \( L = \infty \);
(iii) no conclusion if \( L = 1 \).

Example 1. Determine the convergence of

(a) \( \sum_{n=1}^{\infty} \frac{n}{2^n} \);

(b) \( \sum_{k=0}^{\infty} \frac{k^k}{k!} \).
The Root Test

Given a series \( \sum_{n=1}^{\infty} a_n \) with \( a_n \geq 0 \) for any \( n \). Suppose that \( \lim_{n \to \infty} \sqrt[n]{a_n} = L \). Then

(i) the series converges if \( L < 1 \);
(ii) the series diverges if \( L > 1 \) or \( L = \infty \);
(iii) no conclusion if \( L = 1 \).

Example 2. Determine the convergence of

a. \( \sum_{k=1}^{\infty} \left( \frac{2k+4}{5k+3} \right)^k \);

b. \( \sum_{k=1}^{\infty} \frac{2^k}{k^4} \),
**The Comparison Test:**

Suppose that \(0 \leq a_k \leq b_k\) for all \(k\). Then

1. If \(\sum_{k=1}^{\infty} b_k\) converges, then \(\sum_{k=1}^{\infty} a_k\) converges;
2. If \(\sum_{k=1}^{\infty} a_k\) diverges, then \(\sum_{k=1}^{\infty} b_k\) diverges.

**Example 3.** Investigate the convergence or divergence of

(a) \(\sum_{k=1}^{\infty} \frac{1}{k^3+5k}\);

(b) \(\sum_{k=1}^{\infty} \frac{5^k+1}{2^k-1}\).
The Limit Comparison Test:

Suppose that \( a_k \geq 0 \) and \( b_k \geq 0 \) for all \( k \) such that \( \lim_{k \to \infty} \frac{a_k}{b_k} = c > 0 \), where \( c \) is a constant. Then

either both \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) converge or both \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) diverge.

For some series like \( \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \), it is more convenient to use the limit comparison test than the comparison test.

Example 4. Investigate the convergence or divergence of

(a) \( \sum_{k=1}^{\infty} \frac{1}{2k-1} \);

(b) \( \sum_{n=3}^{\infty} \frac{n^2 - 4n + 7}{n^4 + 5n^2 + 23n - 5} \).
Strategy for testing series

For a given Series $\sum_{k=1}^{\infty} a_k$, we can use the following strategies to test it convergence:

1. If we can see at a glance that $\lim_{k \to \infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ is not convergent.

2. Use special series tests:
   2.1. For geometric series $\sum_{n=0}^{\infty} ar^n$ ($a \neq 0$ a constant), it is convergent for $|r| < 1$ and divergent for $|r| \geq 1$. In order to use the formula to find the sum, $n$ must start from 0.
   2.2 For $p$-series $\sum_{k=1}^{\infty} \frac{a}{k^p}$ ($a \neq 0$ a constant), it is convergent for $p > 1$ and divergent for $p \leq 1$.
   2.3 For telescope series, we can simplify the sum $S_n$ first and then find $\lim_{n \to \infty} S_n$.

3. If $a_k = f(k)$ and $\int_{N}^{\infty} f(x)dx$ is easy to evaluate, then use the integral test.

4. Use ratio test for series if $a_k$ involves with factorial or other product (including a constant raised to the power).

5. Use root test for series if $a_k$ involves with $k$-th power of some numbers.

6. For series looked like $p$-series or geometric series, we can find a $p$-series or a geometric series and then use the comparison or limit comparison test.
8. 6 Alternating Series

There are some series which contain both positive and negative $a_k$'s when $k$ changes. One of such series is called alternating series defined as follows:

$$\sum_{n=1}^{\infty}(-1)^n a_n,$$

where $a_n \geq 0$ or $a_n \leq 0$ for all $n$.

For alternating series we cannot use comparison and limit comparison tests. Why?

The Alternating Series Test

For the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$, if $\lim_{n \to \infty} a_n = 0$ and $0 \leq a_{n+1} \leq a_n$ (nonincreasing) for all $n \geq N$, then it is convergent.

Example 1. Investigate the convergence or divergence of

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ (Alternating Harmonic Series);

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k \ln k}{k}$;

(c) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$. 

Estimating Sums

If $S$ is the sum of an alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ with $\lim_{n \to \infty} a_n = 0$ and $0 \leq a_{n+1} \leq a_n$, then

$$|R_n| = |S - S_n| \leq a_{n+1}.$$ 

Example 2. How many terms are required to approximate the value of $e^{-1} - 1 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$ with the remainder less than $10^{-6}$?
Absolute and Conditional Convergence

**Definition.** Assume a series $\sum_{k=1}^{\infty} a_k$ is convergent. If $\sum_{k=1}^{\infty} |a_k|$ is convergent. Then it is called **absolutely convergent**. Otherwise it is **conditionally convergent**.

**Theorem:** If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is also convergent. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

**Example 3.** Determine the absolute convergence or conditional convergence the following series. If it is convergent, determine whether it is absolutely convergent:

(a) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$;

(b) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$;

(c) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k^3}}$. 

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Practice problems for Test 1:

Determine the convergence or divergence of

(a) \[ \sum_{n=1}^{\infty} \frac{2n-1}{3n+2}, \]

(b) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^3+4n^2+2}. \]

(c) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n+3}, \]

(d) \[ \sum_{k=1}^{\infty} ke^{-k^2}. \]

(e) \[ \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}, \]

(f) \[ \sum_{k=0}^{\infty} \frac{2^k}{k!}. \]

(g) \[ \sum_{n=1}^{\infty} \left( \frac{3n^2+2}{2n^2+3n} \right)^n. \]