PARTITIONS INTO THIN SETS AND FORGOTTEN THEOREMS OF KUNUGI AND LUSIN-NOVIKOV

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Abstract. Let f be a function from a metric space Y to a separable metric space X. If f has the Baire property, then it is continuous apart from a 1st category set. In 1935, Kuratowski asked whether the separability requirement could be lifted. A full scale attack on the problem took place in the late seventies and early eighties. What was not known then, and what remains virtually unknown today, is the fact that a first impressive attempt to solve the Kuratowski problem, due to Kinjiro Kunugi and based on a theorem of Lusin and Novikov, took place already in 1936. Lusin's remarkable 1934 Comptes Rendus note soon forgotten, remained unnoticed to this day. We analyze both papers and bring the results to full light.

1. Introduction

Kazimierz Kuratowski and Pavel S. Alexandrov were friends, both born in 1896, and so in 1976 Kuratowski published, in Russian, an inconspicuous paper [16] dedicated to Alexandrov on his 80th birthday. The paper contained, in particular, the following result.

Proposition 1.1. *Let Y be a complete separable metric space equipped with a non-trivial finite measure μ. Let $\{A_\gamma : \gamma \in \Gamma\}$ be a partition of Y into 1st category (resp., μ-zero) sets. If the Continuum Hypothesis holds, then there exists $\Delta \subset \Gamma$ such that $\bigcup_{\gamma\in\Delta} A_\gamma$ does not have the Baire property (resp., is not μ-measurable).*

Recall that a subset A in a topological space Y is said to be *(of) 1st category* (meager) if it is a countable union of nowhere dense sets; A is said to have the *Baire property* if $A = (O \setminus P) \cup Q$, where O is open and P, Q are 1st category sets. Further, a function f from Y into a topological space X has the Baire property (resp., is measurable with respect to a measure μ on Y) if for each open set $O \subset X$ its inverse image $f^{-1}(O)$ has the Baire property (resp., is μ-measurable) in Y. Some authors, in order to stress the analogy (in the sense of, e.g., [25]) between measure and category, instead of 'has the Baire property', write 'is BP-measurable'.

The result of Kuratowski provoked a flurry of activity. According to an unpublished 1977 manuscript of Prikry (quoted as reference number 27 in [11]), its 'measure part' for the Lebesgue measure on $Y = [0, 1]$ was already known in ZFC to Solovay in 1970. Further, Prikry reports that in that setting, i.e., in ZFC and for $[0, 1]$ with Lebesgue measure, Bukovský got both, the 'measure part' as well as the 'Baire part' using models of set theory, and then follows with a neat analytic proof (discussed in Sectin 6 below) of a generalization of Bukovský's result. Bukovský's research appeared with a considerable delay, in [2]. The same issue of the journal contained another paper, see Emeryk et al. [3], providing a topological proof of the 'Baire part' of the statement with Y being Čech complete of weight less

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than or equal to the continuum. For similar results see also [1], still in the same issue of the journal as [2] and [3].

The reason for the excitement was a 1935 problem [14, Problem 8], and its measure analogue. It was already known that if Y is a metric space, X is a separable metric space, and $f : Y \to X$ has the Baire property, then f is continuous apart from a 1st category set. Kuratowski asked whether that conclusion could be extended to a non-separable framework. Now, again according to the already quoted manuscript of Prikry, Solovay using his result obtained a positive answer to the just mentioned measure analogue of the problem in the following form: *if X is a metric space and $f : [0,1] \to X$ is measurable, then f is almost continuous (Lusin measurable).* It was rather clear that the 'Baire part' ought to provide some answer to the Kuratowski question as well.

As expected, an answer was delivered by Emeryk et al. in [4], a 'sibling' paper of [3] printed next to it, as follows.

Theorem 1.2. *Let Y be a Čech complete space of weight less than or equal to the continuum. If f is a function having the Baire property from Y into a metric space, then f is continuous apart from a 1st category set.*

In the realm of metric spaces the final word concerning Kuratowski's problem is due to Frankiewicz and Kunen [6]. Of course, as one can guess, research continued (see, e.g., [7] and [11]). But what happened later is irrelevant to our story, so we stop at this point.

2. Surprise, surprise

The section above gave a brief history of initial research provoked by Kuratowski's paper, at least history which seems to be currently the accepted one. The initial motivation for the present article came out of an unknown[1] fact that a first attack on the Kuratowski problem, due to Kinjiro Kunugi, happened already in 1936 [12], basing on [20] and [26, Supplément].

In order to be able to discuss that story, we need some more terminology. A subset E of reals $\mathbb{R}$ is said to be an *always (of) 1st category set* if it is 1st category on every perfect set. A subset E of a topological space Y is *2nd category* if it is not 1st category; it is *1st category at (a point) $x \in Y$* if there is an open set (equivalently, a neighborhood [15, Sec. 10, V]) O containing x, such that $E \cap O$ is 1st category. Further, *E is 2nd category at x* if it is not 1st category at x, i.e., if $E \cap V$ is 2nd category for each neighborhood V of x. If K is a subspace of Y and $E \subset K$, then $D_K(E)$ denotes the set of all points x in K such that E is 2nd category in x relative to K. The subscript K is dropped if $K = Y$. If E is 1st (resp., 2nd) category at each point of a set P, we will also say that E *is everywhere 1st* (resp., *2nd*) *category in P*.

Here is the 'Baire part' of Lusin's statement in his 1934 CRAS note [20].

Theorem 2.1. *Let F be a subset of reals which is not always of 1st category. Then F can be written as the union of two disjoint subsets F_1, F_2 that cannot be separated by Borel sets.*

Before going ahead with the proof, let us state an observation:

(0) *Let A and B be subsets of a topological space Y, with $A \subset B$. If A is 2nd category (relative to Y) at each point of B, then A is so relative to B.*

Indeed, suppose to the contrary that there exists an $x \in B$ and an open neighborhood V of x in B such that $V \cap A$ is 1st category relative to B and, therefore, 1st category. Let V^*

[1]Actually, the first author learned about the existence of Kunugi's paper in 1983 from [23]. Somehow, the news remained private and did not spread out despite some efforts to the contrary.

be an open set such that $V^* \cap B = V$. As $V^* \cap A = V \cap A$, we have $x \in V^* \cap B$ and $V^* \cap A$ is 1st category; a contradiction.

Proof. **(1)** Lusin's starting point is the following reduction. Let E be a set which is not always of 1st category (later, in Part 2 of the proof, E appears first as a subset of reals lying on the axis OX in the plane). Then, without loss of generality, we may assume that E is well-ordered into a transfinite sequence $E = \{x_0, x_1, \dots, x_\gamma, \dots\}$ in such a way that every segment of E is an always 1st category set. Moreover, E is contained in a perfect set P such that E is everywhere 2nd category in P relative to P .

Here is the argument. Given F as in the statement of the theorem, let $\{x_0, x_1, \dots, x_\gamma, \dots\}$ be a well-ordering of F. If the requirement about its segments is not satisfied, then find the first index Γ such that $A = \{x_\gamma : \gamma < \Gamma\}$ is not always of 1st category. The latter means that there exists a perfect set K such that $B = A \cap K$ is 2nd category relative to K. According to [15, §10,VI(14)], $B = C \cup E$, where C is 1st category relative to K and E is 2nd category relative to K at each of its points. Let $P = D_K(E)$. Then $E \subset P$ and by [15, §10,V(11)], P is the closure of its interior relative to K, whence perfect. By the very definition of P, the set E is everywhere 2nd category in P relative to K and, by the observation (0) above, relative to P. The well-ordering of E is the one induced from A, assuring that the assumption about segments is satisfied.

(2) By part (1), one has a well-ordered set $E = \{x_0, x_1, \dots, x_\gamma, \dots\}$ contained in a perfect set P; each segment of this transfinite sequence is an always 1st category set, and the set E is everywhere 2nd category in P relative to P.

This was the setting in which Lusin started the proof. Below is our translation of that proof from the French. We keep to the original and its notation closely, including parts that were emphasized by italics.

Locate on the axis OY of the plane XOY a set identical with E and denote by $\mathcal{E}$, where $\mathcal{E} \equiv \{\mathcal{M}(x,y)\}$, the set of points $\mathcal{M}(x,y)$ of the plane such that x and y belong to E and are such that $Ind(y) < Ind(x)$: here Ind means the index (finite or transfinite) of a point in the well-ordered sequence E.

Denote by $\mathcal{E}_{x_0}$ and $\mathcal{E}_{y_0}$ two linear sets of all points of $\mathcal{E}$ located, respectively, on the lines $x = x_0$ and $y = y_0$. It is clear that $\mathcal{E}_x$ is an always 1st category set for each x. Thus, we can write

$$\mathcal{E} = \bigcup_{x\in E} \mathcal{E}_x = \bigcup_{x\in E} \bigcup_{n=1}^{\infty} \mathcal{E}_x^{(n)} = \bigcup_{n=1}^{\infty} \bigcup_{x\in E} \mathcal{E}_x^{(n)} = \bigcup_{n=1}^{\infty} H_n,$$

where $\mathcal{E}_x^{(n)}$ is nowhere dense in P (parallel to OY). In contrary, $\mathcal{E}_y$ is evidently everywhere 2nd category in P relative to P (parallel to OX) when $y \in E$.

It follows that, if $\pi_1, \pi_2 \dots \pi_i \dots$ is a sequence formed by subsets of P determined by the intervals with rational end points, for each $y \in E$ there corresponds a pair of positive integers (n, i) such that $(H_n)_y$ is everywhere 2nd category in π_i (parallel to OX). Let $E_{n,i}$ be the set of points $y \in E$ corresponding to the same pair (n, i). As there is a countable number of the sets $E_{n,i}$ and as E (on OY) is 2nd category in P, one sees easily that there exists a pair (ν, ι) such that $E_{\nu,\iota}$ is dense in some π_j (on OY).

Now, $(H_\nu)_x$ is nowhere dense in P (parallel to OY) whatever $x \in E$. Thus, for each $x_0 \in E$ there corresponds positive integer k such that π_k in P (on $x = x_0$) contained in π_j does not contain any point of $\pi_k \cap (H_\nu)_{x_0}$. Denote by $E^{(k)}$ the set of points $x \in E$ that correspond to the same number k. As there is countable number of $E^{(k)}$ and as E is everywhere 2nd category in P, there exists π_μ in P (on OX) contained in π_ι and such that $E^{(\kappa)}$ *is everywhere 2nd category in* π_μ.

But there surely exists a point $y_0 \in E_{\nu,\iota}$ that belongs to π_κ (on OY). It is clear that $(H_\nu)_{y_0}$ is everywhere 2nd category in π_ι and, consequently, *is everywhere 2nd category in* π_μ.

In the end, *we have found two subsets,* E_1 *and* E_2 *of* E *that are disjoint and such that each one is everywhere 2nd category in* π_μ.

This was the end of the original proof. Lusin did not name E_1 and E_2, but these are $E^{(\kappa)}$ and $(H_\nu)_{y_0}$ that was moved from the line $y = y_0$ onto OX.

(3) By part (1) of the proof, $F = F_0 \cup E$ with $F_0 \cap E = \emptyset$, and $E \subset P$, where P is a perfect set such that E satisfies the conditions specified in (2). As $E \subset P$ and Borel sets relative to P are traces of Borel sets, it is sufficient to consider P as the enveloping space. Let E_1 and E_2 be subsets of E found in (2). Suppose there exists a Borel subset B of P providing separation, i.e., let $B \supset E_1$ and B be disjoint with E_2. Clearly, B is 2nd category at each point of π_μ and its complement $P \setminus B$, containing E_2, is also 2nd category at each point of π_μ. Yet, in view of [15, §11, IV], B cannot even have the Baire property relative to P and, a fortiori, cannot be a Borel subset of P. A contradiction. Consequently, E_1 and E_2 cannot be separated by Borel sets and the same is true for $F_1 = F_0 \cup E_1$ and $F_2 = F \setminus F_1$. □

Remark 2.2. Lusin treated as evident not only part (1), but also part (3) above; only part (2) was given in the Note.

Kunugi, who was after Kuratowski's problem, imposed at the beginning of his paper [12] the following *condition* (α) on a space Y:

Given an ordinal number γ*, let* $\{A_\xi\}$ *be a disjoint family of 1st category subsets of* Y *with* ξ *running through all ordinals less than* γ*. If the union* $\bigcup_{\xi<\gamma} A_\xi$ *is 2nd category, then one can break the union into two disjoint parts* $\bigcup_{\xi'} A_{\xi'}$ *and* $\bigcup_{\xi''} A_{\xi''}$ *in such a way that each part is everywhere 2nd category in a common open subset of* Y.

For a proof that (α) holds in a separable metric space, he referred to Lusin [20] and to Sierpiński [26, Supplément].

Now Sierpiński, referring to a private communication from Lusin, stated in his Supplément the following 'Lemme'.

Lemma 2.3. *Let* E *be a 2nd category subset of an interval* I*. There exist an interval* $J \subset I$ *and two disjoint subsets* E_1 *and* E_2 *of* E *that are 2nd category at each point of* J.

Lemma 2.3 ought to be compared with the part (2) of the proof of Theorem 2.1. Sierpiński's assumption is stronger, and the conclusion too, because Lusin's conclusion is 'at each point of $J \cap P$' for some perfect set P, and Sierpiński has 'at each point of J'.

The condition (α) *of Kunugi is nothing else, but Lemma 2.3 (in the realm of separable metric spaces), in which points of the set* E *are replaced by disjoint subsets of 1st category.*

Whether Kunugi thought that the passage from 'points' to 'disjoint 1st category subsets' posed no problem, or knew how to accomplish it and yet did not bother to give a proof, we will never know. What is not in doubt, is the fact that it was the 'Lemme' in [26], which prompted him to formulate the condition (α).

We mention this, because we first considered Sierpiński's proof, which is rather long and does not bear any resemblance with the arguments in (2) above. It was not visible how to generalize it, to get (α). Only then, we focused on Lusin's Note, despite the fact that a direct translation of the result there would not produce the condition (α).

It turned out that the technique developed in (2) permits it to obtain Lemma 2.3 as well. Indeed, let us replace the assumption 'F is not always 1st category' in Theorem 2.1, by 'F is 2nd category'. The reduction described in part (1) of its proof can now be performed without the intermediate step involving K. Hence, one ends up with $E \subset P$ and $P = D(E)$. It follows that the set π_μ appearing at the end of Lusin's proof is a non-empty set of the form $(a,b) \cap D(E)$. It must contain an interval J, since $D(E)$ is the closure of its non-empty interior ([15, §10,V(11)]), which gives Sierpiński's conclusion.

3. Disjoint families of 1st category sets

An examination of the technique used in the proof of Theorem 2.1 allows us to prove the following result, which confirms that Kunugi's claim about the validity of the condition (α) was correct. The relevant topological fact needed for the proof is the existence of a countable base for open sets, i.e., that the space is *2nd countable*.

Theorem 3.1. *Let $\{A_\gamma : \gamma \in \Gamma\}$ be a family of disjoint 1st category subsets of a 2nd countable topological space Y. If the union $E = \bigcup_{\gamma\in\Gamma} A_\gamma$ is 2nd category, then there exist $\Delta \subset \Gamma$ and an open set O such that $\bigcup_{\gamma\in\Delta} A_\gamma$ and $\bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma$ are everywhere 2nd category in O. In consequence, $\bigcup_{\gamma\in\Delta} A_\gamma$ and $\bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma$ cannot be separated by sets having the Baire property.*

Proof. We may assume without loss of generality (cf. Part 1 in the proof of Theorem 2.1) that Γ is an ordinal, $\gamma < \Gamma$, and that for each $\gamma < \Gamma$, the set $\bigcup_{\beta<\gamma} A_\beta$ is of 1st category.

Consider the product $Y \times Y$. It will be convenient to speak in terms of the XOY coordinate system with the first copy of Y as the 'horizontal x-axis', and the second copy of Y as the 'vertical y-axis'. For any $\beta < \Gamma$, set $B_\beta = \bigcup_{\gamma<\beta} A_\gamma$. Define $\mathcal{E} \subset Y \times Y$ by

$$\mathcal{E} = \bigcup_{\gamma<\Gamma} (A_\gamma \times B_\gamma).$$

The 'vertical' set $\mathcal{E}_{x_0}$ lying on the 'line $x = x_0$' (i.e., on $\{(y, x_0) : y \in Y\}$) is 1st category for every $x_0 \in Y$. Write

$$\mathcal{E} = \bigcup_{x\in E} \mathcal{E}_x = \bigcup_{\gamma<\Gamma} \bigcup_{x\in A_\gamma} (\{x\} \times B_\gamma) = \bigcup_{\gamma<\Gamma} \bigcup_{n=1}^{\infty} (A_\gamma \times B_\gamma^{(n)}) = \bigcup_{n=1}^{\infty} H_n,$$

where $H_n = \bigcup_{\gamma<\Gamma}(A_\gamma \times B_\gamma^{(n)})$ and the sets $B_\gamma^{(n)}$ are nowhere dense in Y.

Let

$$O_1, O_2 \ldots O_i \ldots$$

be a countable base of open sets of Y. For each $y_0 \in E$, one can find α such that $y_0 \in A_\alpha$. Then, $y \in B_\beta$ for any $\beta > \alpha$ and $\mathcal{E}_{y_0} = \bigcup_{\alpha<\beta<\Gamma} A_\beta$. In particular, the 'horizontal' set $\mathcal{E}_y$ lying on the 'line $y = y_0$' (i.e., on $\{(x, y_0) : x \in Y\}$) is 2nd category for each $y_0 \in E$.

As $\mathcal{E}_y = (\bigcup_{n=1}^{\infty} H_n)_y$ is 2nd category, there exists $n \in \mathbb{N}$ so that $(H_n)_y$ is 2nd category and so, by [15, §10 (7) and (11)], $IntD((H_n)_y)$ is non-empty. It follows that for every $y \in E$, one can find a pair (n,i) of naturals such that $(H_n)_y$ is everywhere 2nd category in O_i ('parallel' to the x-axis). Let

$$(*) \qquad E_{n,i} = \{y \in E : (H_n)_y \text{ is everywhere 2nd category in } O_i\},$$

i.e., the set of all points $y \in E$ corresponding to the same pair (n,i).

As $E = \bigcup\{E_{n,i} : n \in \mathbb{N}, i \in \mathbb{N}\}$ and E (on the y-axis) is 2nd category, there exists a pair (ν, ι) such that $E_{\nu,\iota}$ is dense in some O_j (on the y-axis).

Further, as $(H_\nu)_x$ is nowhere dense in Y ('parallel' to the y-axis) for any $x \in E$, it follows that for each $x_0 \in E$ there exist a natural number k and O_k (on the line $x = x_0$) contained in O_j (moved from the y-axis onto the line $x = x_0$) such that $O_k \cap (H_\nu)_{x_0} = \emptyset$.

Denote by $E^{(k)}$ the set of points $x \in E$ that correspond to $k \in \mathbb{N}$. If $y \in E_{\nu,\iota}$, in view of $(*)$, $(H_\nu)_y$ is everywhere 2nd category in O_ι. A fortiori $(E)_y$ is so there too. But $E = \bigcup_{k=1}^\infty E^{(k)}$, so there must exist O_μ (on the x-axis) contained in O_ι and κ such that $E^{(\kappa)}$is everywhere 2nd category in O_μ.

Now suppose $x_1 \in E^{(\kappa)}$ and x_2 is such that x_1, x_2 belong to the same A_γ. Then $O_\kappa \cap H_\nu = \emptyset$ on $x = x_1$ implies the same on $x = x_2$ and therefore $x_2 \in E^{(\kappa)}$, i.e., $E^{(\kappa)}$ is *saturated* (in the sense that it defines $\Delta \subset \Gamma$ such that $E^{(\kappa)} = \bigcup_{\gamma\in\Delta} A_\gamma$).

But, as $E_{\nu,\iota}$ was dense in O_j, there surely exists a point $y_0 \in E_{\nu,\iota}$ which belongs to O_κ (on the y-axis). Since $(H_\nu)_{y_0}$ is everywhere 2nd category in O_ι, it is so in O_μ as well.

Again, suppose that $(x_1, y_0) \in (H_\nu)_{y_0}$ and x_2 belongs to the same A_γ as x_1. Then, from the definition of the sequence (H_n), also $(x_2, y_0) \in (H_\nu)_{y_0}$. Hence $(H_\nu)_{y_0}$ is saturated and therefore defines $\Delta' \subset \Gamma$ such that $\bigcup_{\gamma\in\Delta'} A_\gamma$ (moved on the line $y = y_0$) equals $(H_\nu)_{y_0}$. By construction, $E^{(\kappa)}$ and $(H_\nu)_{y_0}$ (moved from the line $y = y_0$ onto the x-axis) are disjoint and, consequently, also Δ and Δ' are disjoint. Moreover, the corresponding unions are everywhere 2nd category in O_μ. It is also clear that $\bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma$, containing $\bigcup_{\gamma\in\Delta'} A_\gamma$, must be everywhere 2nd category in O_μ. Set $O_\mu = O$.

Finally, observe that there is no $G \subset Y$ having the Baire property such that $G \supset \bigcup_{\gamma\in\Delta} A_\gamma$ and $G \cap \bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma = \emptyset$. Indeed, suppose the contrary. Then G, containing $\bigcup_{\gamma\in\Delta} A_\gamma$, must be everywhere 2nd category in O and its complement G', containing $\bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma$, is also everywhere 2nd category in O. Hence, in view of [15, §11, IV], G cannot have Baire property. A contradiction. In particular, $\bigcup_{\gamma\in\Delta} A_\gamma$ does not have the Baire property. □

Corollary 3.2. *Let E be a 2nd category subset of a 2nd countable T_1-space Y without isolated points. Then E can be written as the union of two disjoint subsets that cannot be separated by sets having the Baire property.*

Proof. A topological space is T_1 if its one-point subsets are closed. Thus, if a singleton is not an isolated point, then it is nowhere dense, and we can apply the above Theorem for a partition of E into points. □

According to [15, §40, II], a separable metric space Y is said to be an *always of 1st category* space if every dense in itself subset of Y is 1st category in itself. A subset in a *Polish space* (i.e., a 2nd countable topological space that admits a complete metric) Y is always of 1st category iff it is 1st category on every perfect subset of Y [15, §40, II, Theorem 1]. Consider also the σ-algebra of sets having the Baire property with the restricted sense ([15, §11, VI], that is, sets whose traces on every subset of Y have the Baire property relative to that subset. The next result is a generalization of Theorem 2.1 to partitions.

Theorem 3.3. *Let $\{A_\gamma : \gamma \in \Gamma\}$ be a family of disjoint always 1st category subsets of a Polish space Y. If the union $E = \bigcup_{\gamma\in\Gamma} A_\gamma$ is not an always 1st category subset of Y, then there exists $\Delta \subset \Gamma$ such that $\bigcup_{\gamma\in\Delta} A_\gamma$ and $\bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma$ cannot be separated by sets having the Baire property in the restricted sense.*

Proof. If E is not always of 1st category, then there exists a perfect subset K of Y such that $E \cap K$ is 2nd category relative to K. Consider the family $\{A_\gamma \cap K : \gamma \in \Gamma\}$, and note that it is a family of 1st category subsets of K. By applying Theorem 3.1, we will find $\Delta \subset \Gamma$ such that $\bigcup_{\gamma\in\Delta} A_\gamma \cap K$ and $\bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma \cap K$ cannot be separated by sets having the Baire property relative to K. Now, suppose there exists a set C containing $\bigcup_{\gamma\in\Delta} A_\gamma$, which is disjoint with the union of A_γ's over $\Gamma \setminus \Delta$ and, moreover, has the Baire propewrty in the resticted sense. Then $C \cap K$ contains $\bigcup_{\gamma\in\Delta} A_\gamma \cap K$, is disjoint with $\bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma \cap K$ and, therefore, separates these two unions on K. This contradicts Theorem 3.1, because $C \cap K$ has the Baire property relative to K. □

Corollary 3.4. *Let Y be a Polish space without isolated points and $E \subset Y$. If E is not an always 1st category set, then it can be written as the union of two disjoint sets that cannot be separated by sets having the Baire property in the restricted sense.*

4. Kunugi's Theorem

Theorem 4.1. *Let X be a topological space satisfying the condition (α), Y a metric space, and $f : X \to Y$ a function having the Baire property. Then f is continuous apart from a 1st category set.*

Below is Kunugi's proof. We keep to the original and its notation closely. Although a few references to [15, §§10 and 11] would be helpful (at that time it would be [13], to which in fact Kunugi referred while giving the definition of the Baire property), the proof is totally 'modern', correct, and we think – quite impressive, taking into account the date of its discovery.

Proof. Kunugi calls a set to be of type G_ρ if it is a set-theoretical difference of two open sets. He begun his proof by stating:

As $Y = (Y, \rho)$ is a metric space, by a result of Montgomery [22], it is possible to find a sequence $Y^n\,(n = 1, 2, 3, \dots)$ such that $Y = \bigcup_{n=1}^\infty Y^n$ and each Y^n decomposes into a transfinite sequence Y^n_ξ of G_ρ-sets with distance $\rho(Y^n_\xi, Y^n_{\xi'}) > \frac{1}{n}$ for $\xi \neq \xi'$. Besides, we can assume that given a base of neighborhoods for Y, Y^n_ξ is contained in some of those neighbourhoods whatever n and ξ.

The above statement is correct. However, its derivation from [22] *being not quite immediate, we provide a proof in the Appendix (Section 7 below). Kunugi continues*:

Letting (ε_k) to be a sequence of positive reals converging to 0, we assume that the diameter of Y^n_ξ is less than ε_k. Set $Y = \bigcup_{n=1}^\infty Y^n(\varepsilon_k)$ and $Y^n(\varepsilon_k) = \bigcup_\xi Y^n_\xi(\varepsilon_k)$.

By our assumption, $f^{-1}(Y^n_\xi(\varepsilon_k))$ has the Baire property. We can thus write

$$f^{-1}(Y^n_\xi(\varepsilon_k)) = (G^n_\xi(k) \setminus P^n_\xi(k)) \cup Q^n_\xi(k),$$

where $G^n_\xi(k)$ is open, while $P^n_\xi(k)$ and $Q^n_\xi(k)$ are 1st category sets.

Write $D(n, k) = \bigcup_\xi (P^n_\xi(k) \cup Q^n_\xi(k))$, and assume that we can find $P^n_\xi(k)$ and $Q^n_\xi(k)$ in such a way that $D(n, k)$ is 1st category for all n and k. We are going to show that the function f is continuous when one neglects the set $D = \bigcup_n \bigcup_k D(n, k)$. Indeed, let G be an open subset of Y. Consider $\bigcup' Y^n_\xi(\varepsilon_k)$, where the union $\bigcup'$ is taken over all n, k such that $Y^n_\xi(\varepsilon_k) \subset G$. For a point $p \in G$ find a ball of radius ε_k contained in G. Then, there exist n and ξ such that $p \in Y^n_\xi(\varepsilon_k) \subset G$. Hence $G = \bigcup' Y^n_\xi(\varepsilon_k)$. It follows that $f^{-1}(G) \cap (X \setminus D) = \bigcup' G^n_{\xi(k)} \cap (X \setminus D)$. Hence $f^{-1}(G)$ is open in $X \setminus D$.

We have shown that f is continuous apart from 1st category set D. It remains to prove that it is possible to choose the sets $P^n_\xi(k)$ and $Q^n_\xi(k)$ in such a way that $D(n, k)$ be of 1st category.

Let us now say that a transfinite sequence of disjoint sets having the Baire property

$$(*) \qquad X_0, X_1, X_2 \dots X_\xi \dots$$

satisfies the condition (P), if we can write $X_\xi = (G_\xi \setminus P_\xi) \cup Q_\xi$, with 1st category P_ξ and Q_ξ, so that the union $\bigcup_\xi (P_\xi \cup Q_\xi)$ be of 1st category. Moreover, the sequence $(*)$ *satisfies the condition* (P) *at a point* p, if there exists a neighborhood $V(p)$ of p, such that the sequence

$$X_0 \cap V(p), X_1 \cap V(p), \dots X_\xi \cap V(p) \dots$$

satisfies the condition (P). The set of all points of X at which the sequence $(*)$ satisfies (P) will be denoted $\{X_\xi\}_I$, and $X \setminus \{X_\xi\}_I$ will be denoted $\{X_\xi\}_{II}$. Now, one can see without much pain that

(I) if $\{X_\xi\}_{II} = \emptyset$, the sequence $(*)$ satisfies the condition (P);
(II) if $\{X_\xi\}_{II} \neq \emptyset$, $\{X_\xi\}_{II}$ is 2nd category at each point of $\{X_\xi\}_{II}$;
(III) $\{X_\xi\}_{II}$ is the closure of an open set.

Suppose now $\{X_\xi\}_{II} \neq \emptyset$. By (III), there exists an open set G such that $\{X_\xi\}_{II} = \overline{G}$. We claim that $G \cap X_\xi$ is 1st category for all ξ. Indeed, otherwise $G \cap X_{\xi_0}$ is 2nd category for some ξ_0. Consequently, there exist $p \in G$ and a neighborhood $V(p) \subset G$ of p such that $G \cap X_{\xi_0}$ is 2nd category at each point of $V(p)$. X_{ξ_0} having the Baire property, $V(p) \setminus X_{\xi_0}$ is 1st category. As $\bigcup_{\xi \neq \xi_0} X_\xi$ is disjoint with X_{ξ_0}, one has $\bigcup_{\xi \neq \xi_0} V(p) \cap X_\xi \subset V(p) \setminus X_{\xi_0}$, and therefore is 1st category. Defining $G_{\xi_0} = V(p), P_{\xi_0} = V(p) \setminus X_{\xi_0}$, $Q_{\xi_0} = \emptyset$, and $G_\xi = P_\xi = \emptyset$, $Q_\xi = V(p) \cap X_\xi$ for $\xi \neq \xi_0$, we see that the sequence $(X_\xi \cap V(p))$ satisfies the condition (P) at p, in contradiction with the choice of the point p.

It follows that if $\{X_\xi\}_{II} \neq \emptyset$, then there is a non-empty open set G such that (i) $G \cap X_\xi$ is 1st category for each ξ; (ii) $G \cap X_\xi$ are disjoint; (iii) $G \cap \bigcup_\xi X_\xi$ is everywhere 2nd category in G. If so, our assumption allows us to break the sequence $(*)$ into two disjoint pieces: $\bigcup_{\xi'} X_{\xi'} \cap G$ and $\bigcup_{\xi''} X_{\xi''} \cap G$ in such a way that they are everywhere 2nd category in an open set. Thus, the set $\bigcup_{\xi'} X_{\xi'}$ does not have Baire property.

If we define $X_\xi = f^{-1}(Y_\xi^n(\varepsilon_k))$, and if $\{X_\xi\}_{II} \neq \emptyset$, as the set $\bigcup_{\xi'} Y_{\xi'}^n(\varepsilon_k)$ is of type G_ρ, the function f would not have the Baire property, in contradiction with our hypothesis.

We see, therefore, that $\{f^{-1}(Y_\xi^n)(\varepsilon_k)\}_{II} = \emptyset$. This means, by (I), that the transfinite sequence $(f^{-1}(Y_\xi^n)(\varepsilon_k))_\xi$ satisfies the condition (P), whatever n and k $(n, k = 1, 2, 3, \dots)$. □

5. Disjoint families of measure zero sets

The second part of [20] deals with a measure analogue of the first part. Instead of 2nd category subsets of reals, the sets of positive outer Lebesgue measure m_e are considered. Lusin's initial reduction corresponds to the step (1) in the proof of Theorem 2.1. He then ends up with a well-ordered subset $E = \{x_0, x_1, \dots, x_\gamma, \dots\}$ of a perfect set P such that $m_e(E) = m(P) < \infty$ and every segment of E is of measure zero. The definition of the 'matrix' $\mathcal{E} \equiv \{\mathcal{M}(x, y)\}$ remained the same and the proof, which we will now adapt for partitions, followed.

Again, the existence of a countable base of open sets was essential. Consequently, as in Section 3, Y *is assumed to be 2nd countable.* Our measure m is *the completion of a finite regular Borel measure* on Y and its *outer measure* is denoted by m_e.

The passage from 'points' to 'disjoint measure zero sets' can be done without much pain. As in the 'Baire part', sets $B_\gamma = \bigcup_{\alpha \leq \gamma} A_\alpha$ for $\gamma < \Gamma$ need to be defined and, instead of the matrix $\mathcal{M}(x,y)$, one has to consider its analogue in which points x_γ are replaced by sets A_γ. Once this done, the proof below is a modification of the proof in [20].

We add some auxiliary results in order to make the presentation reasonably accessible. For the upcoming lemma see, for instance, [10, § 12].

Lemma 5.1. *Let $A \subset B$ be subsets of Y, the set B being measurable. The following conditions are equivalent.*

(a) $m_e(A) = m(B)$.
(b) *If G is measurable and $G \subset B \setminus A$, then $m(G) = 0$.*

A set B described by the above lemma is called an (m-measurable) *envelope* of A; it is often denoted by $\tilde{A}$. Note that $m(\tilde{A}) = \inf\{m(C) : A \subset C\}$, where the sets C run over measurable sets.

The following lemma ([10, § 12(4)]) can be obtained as an easy application of the notion of envelope.

Lemma 5.2. *If $A_1 \subset A_2 \subset \dots$ is a sequence of subsets of Y, then $m_e(A_n) \uparrow m_e(\bigcup_{n=1}^\infty A_n)$.*

The next lemma is [9, Proposition 6.1.322] or [24, Proposition 11.2.5].

Lemma 5.3. *Let (A_n) be a sequence of subsets of Y. Suppose that there exist disjoint measurable sets B_n such that $A_n \subset B_n$ for each $n \in \mathbb{N}$. Then $m_e(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty m_e(A_n)$.*

Proof. One can assume that $B_n = \tilde{A}_n$, an envelope of A_n, for every $n \in \mathbb{N}$. We then have $m_e(\bigcup_n A_n) = m(\bigcup \tilde{A}_n)$. Indeed, let G be measurable and $G \subset (\bigcup_n \tilde{A}_n \setminus \bigcup_n A_n)$. Then $G \cap \tilde{A}_n$ is a measurable set contained in $\tilde{A}_n \setminus A_n$ and therefore, by Lemma 5.1, it is of measure zero. Hence $G = \bigcup_n G \cap \tilde{A}_n$ is also of measure zero. Applying Lemma 5.1 again, $m_e(\bigcup_n A_n) = m(\bigcup_n \tilde{A}_n) = \sum_n m(\tilde{A}_n) = \sum_n m_e(A_n)$. □

Corollary 5.4. *Suppose (A_n) is a sequence of subsets of Y that can be pairwise separated by measurable sets. Then $m_e(\bigcup_n A_n) = \sum_n m_e(A_n)$.*

Proof. By our assumption, for each $n \neq m$, there exists a measurable set B_{nm} such that one has $A_n \subset B_{nm}$ and $B_{nm} \cap A_m = \emptyset$. For $n \in \mathbb{N}$, write $C_n = \bigcap\{B_{nm} : m \in \mathbb{N}, m \neq n\}$. Then, for each n, we have

$$(+) \qquad A_n \subset C_n \text{ and } C_n \cap A_m = \emptyset \text{ for } m \neq n,$$

and C_n are measurable. Define $B_n = C_n \setminus \bigcup_{k=1}^{n-1} C_k$. It follows from (+) that the B_n's satisfy the assumption of Lemma 5.3. □

Denoting by $m \times m$ the product measure in $Y \times Y$, we have

Lemma 5.5. *Let $Z \subset Y \times Y$, and a, b be positive reals. If $m_e(\{y : m_e(Z_y) > b\}) > a$, then $(m \times m)_e(Z) > ab$.*

Proof. Let B be a measurable (with respect to the product σ-algebra) set containing Z such that $(m \times m)(B) = (m \times m)_e(Z)$. We therefore have $m(\{y : m(B_y) > b\}) > a$. By the Fubini theorem,

$$(m \times m)(B) = \int_{Y \times Y} 1_B \, d(m \times m)(x,y) = \int_Y \left(\int_Y 1_B(x,y) \, dm(x) \right) dm(y) = \int_Y f(y) \, dm(y),$$

where $m(\{y : f(y) > b\} > a$. So $\int_Y f(y) \, dm(y) > ab$ and $(m \times m)_e(Z) = (m \times m)(B) > ab$. □

Lemma 5.6. *Let Z_1 and Z_2 be subsets of Y. Then*

$$(m \times m)_e(Z_1 \times Z_2)_e = m_e(Z_1)m_e(Z_2).$$

Proof. If $C \subset Y \times Y$ is measurable containing $Z_1 \times Z_2$ and $(m \times m)(C) = m_e(Z_1 \times Z_2)$, then $m_e(Z_1)m_e(Z_2) = m(\tilde{Z}_1)m(\tilde{Z}_2) = (m \times m)(\tilde{Z}_1 \times \tilde{Z}_2) \geqslant (m \times m)_e(Z_1 \times Z_2) = (m \times m)(C)$. But, by Lemma 5.5, $(m \times m)(C) \geqslant m_e(Z_1)m_e(Z_2)$ and the asserted equality follows. □

Theorem 5.7. *Let $\{A_\gamma : \gamma \in \Gamma\}$ be a disjoint family of measure zero subsets of a 2nd countable topological space Y and $E = \bigcup_{\gamma\in\Gamma} A_\gamma$. If $m_e(E) > 0$, then there exists $\Delta \subset \Gamma$ such that $\bigcup_{\gamma\in\Delta} A_\gamma$ and $\bigcup_{\gamma\in\Gamma\setminus\Delta} A_\gamma$ cannot be separated by measurable sets.*

Proof. Similarly as before, we may assume without loss of generality that Γ is an ordinal, $\gamma < \Gamma$, and that for each $\gamma < \Gamma$, one has $m(\bigcup_{\beta<\gamma} A_\beta) = 0$.

Let $\{O_n : n \in \mathbb{N}\}$ be a base of open sets in Y and, for each $\gamma < \Gamma$, let $B_\gamma = \bigcup_{\alpha\leq\gamma} A_\alpha$. As $m(B_\gamma) = 0$, we can find basic open sets O_n^γ such that $B_\gamma \subset \bigcup_{n=1}^\infty O_n^\gamma$ and $m(\bigcup_{n=1}^\infty O_n^\gamma) < \varepsilon$, where $\varepsilon > 0$ and is as small as we wish. Using the conventions of the proof of Theorem 3.1, write

$$\mathcal{E} = \bigcup_{x\in E} \mathcal{E}_x = \bigcup_{\gamma<\Gamma} \bigcup_{x\in A_\gamma} (\{x\} \times B_\gamma) = \bigcup_{\gamma<\Gamma} \bigcup_{n=1}^\infty (A_\gamma \times B_\gamma^{(n)}) = \bigcup_{n=1}^\infty H_n,$$

where $H_n = \bigcup_{\gamma<\Gamma}(A_\gamma \times B_\gamma^{(n)})$ and $B_\gamma^{(n)} = O_n^\gamma \cap B_\gamma$.

For each $y \in E$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$m_e\left((\bigcup_{i=1}^n H_i)_y\right) > m_e(E) - \varepsilon.$$

Indeed, $(\bigcup_{i=1}^n H_i)_y \uparrow (\bigcup_{i=1}^\infty H_i)_y = \mathcal{E}_y$, whence by Lemma 5.2,

$$m_e\left((\bigcup_{i=1}^n H_i)_y\right) \uparrow m_e\left((\bigcup_{i=1}^\infty H_i)_y\right) = m_e(\mathcal{E}_y) = m_e(E).$$

Consequently, by applying Lemma 5.2 again, there exists $N \in \mathbb{N}$ such that

$$(*) \qquad m_e\left(\{y : m_e((\bigcup_{i=1}^N H_i)_y) > m_e(E) - \varepsilon\}\right) > m_e(E) - \varepsilon.$$

In order to see this, let $A^N = \{y : m_e((\bigcup_{i=1}^N H_i)_y) > m_e(E) - \varepsilon\}$. Clearly, $\bigcup_{N=1}^\infty A^N = E$, and so $m_e(A^N) \uparrow m_e(E)$ with $N \to \infty$.

For N just found, denote $\mathcal{H} = \bigcup_{i=1}^N H_i$. From $(*)$ and Lemma 5.5, we have

$$(**) \qquad (m \times m)_e(\mathcal{H}) > (m_e(E) - \varepsilon)^2.$$

By the definitions of $\mathcal{H}$ and $(O_n^\gamma)_{n=1}^\infty$, for each $x \in A_\gamma$ with $\gamma < \Gamma$ one gets

$$\mathcal{H}_x \subset O_1^\gamma \cup O_2^\gamma \cup \ldots O_N^\gamma.$$

For each sequence $(U_1, U_2, \ldots U_N)$ of basic open sets, define

$$E^{U_1U_2\ldots U_N} = \bigcup\{A_\gamma : (O_1^\gamma, O_2^\gamma, \ldots O_N^\gamma) = (U_1, U_2, \ldots U_N)\}.$$

These are disjoint sets. Ordering those that are non-empty into a sequence, one gets

$$E^1, E^2, \ldots E^k \ldots$$

Now, $E = \bigcup_k E^k$ and we claim that the sets E^k cannot be separated by measurable sets. Indeed, if the separation were possible, by Corollary 5.4 one would have the equality

$$m_e(E) = \sum_{k=1}^{\infty} m_e(E^k).$$

But

$$\mathcal{H} \subset \bigcup_{k=1}^{\infty} (E^k \times \{U_1^k \cup U_2^k \cup \dots U_N^k\}),$$

where $E^k = E^{U_1^k U_2^k \dots U_N^k}$. Moreover, one has

$$\begin{aligned}(m \times m)_e(\mathcal{H}) &\leqslant \sum_k (m \times m)_e(E^k \times \{U_1^k \cup U_2^k \cup \dots U_N^k\}) \\ &= \sum_k m_e(E^k) m(\{U_1^k \cup U_2^k \cup \dots U_N^k\}) < \varepsilon \sum_k m_e(E^k),\end{aligned}$$

where the equality in the middle holds by Lemma 5.6. Hence

$$(m \times m)_e(\mathcal{H}) \leqslant \varepsilon \cdot m_e(E),$$

which gives a contradiction with $(**)$, in view of the arbitrariness of ε.

There must exist, therefore, indices i and l such that E^i and E^l cannot be separated by measurable sets. A fortiori, E^i and $\bigcup_{k \neq i} E^k$ cannot be so separated. Let Δ be the set of γ's determined by A_γ's corresponding to E^i. □

In particular, if $m(\bigcup_{\gamma \in \Gamma} A_\gamma) > 0$, then there exists $\Delta \subset \Gamma$ such that $\bigcup_{\gamma \in \Delta} A_\gamma$ is not measurable.

Corollary 5.8. *Let E be a subset of a 2nd countable T_1-space Y equipped with the completion of a finite regular Borel measure m vanishing on points. If $m_e(E) > 0$, then E can be written as the union of two disjoint subsets that cannot be separated by measurable sets.*

Proof. One point subsets of Y, being closed, are in the domain of the measure m and so of measure zero. We may apply the above Theorem to the partition of E into points. □

A subset E of Y is said to be *universally measurable* with respect to a non-empty family of measures on Y if it is measurable with respect to each measure μ in the family; it is *universally null* with respect to this family if $\mu(E) = 0$ for each measure μ in the family. Let $\mathcal{M}$ be a family of the completions of regular Borel measures on Y. One shows easily the following measure analogue of Theorem 3.3.

Theorem 5.9. *Let $\{A_\gamma : \gamma \in \Gamma\}$ be a family of disjoint universally $\mathcal{M}$-null subsets of a 2nd countable topological space. If $E = \bigcup_{\gamma \in \Gamma} A_\gamma$ is not universally $\mathcal{M}$-null, then there exists $\Delta \subset \Gamma$ such that $\bigcup_{\gamma \in \Delta} A_\gamma$ and $\bigcup_{\gamma \in \Gamma \setminus \Delta} A_\gamma$ cannot be separated by universally $\mathcal{M}$-measurable sets.*

Denote by $\mathcal{N}$ the family of the completions of regular Borel measures, defined on a topological T_1-space, such that each measure vanishes on points.

Corollary 5.10. *Let Y be a 2nd countable topological T_1-space and let E be a subset of Y that is not universally $\mathcal{N}$-null. Then it can be written as the union of its two disjoint subsets that cannot be separated by universally $\mathcal{N}$-measurable sets.*

The CRAS note [20] (Séance de 23 avril) was preceded by two other notes by Lusin, [18] (Séance de 19 mars) and [19] (Séance de 26 mars). In those notes Lusin discussed a new method of solving four 'difficiles' (difficult) problems 'de la théorie des fonctions' (of the theory of functions). In [19], he thought he had solved Problems 1 and 2 on the list, by showing that *every uncountable subset of reals can be decomposed into two disjoint subsets that cannot be separated by Borel sets*[2]. He mentioned in a footnote that Pyotr S. Novikov also believed having a solution, and indicated that he did not know Novikov's proof. In the introductory Section 1 of the note [20] that we study in this paper, Lusin admitted an error in both proofs – his and Novikov's. He wrote (that in this situation) "il me semble utile de développer complètement la démonstration dans ces cas importants où" (it seems useful to me to give a full proof in those important cases when) *the set is not always 1st category or is not of measure zero.* The next sentence is: "Je suivrai de près la manière de Monsieur P. Novikoff." In English: "I will follow the technique of Mr. P. Novikov closely." Thus, it appears that Lusin was able to present proofs of Theorem 2.1 and Corollary 5.8 using the technique that he learned in the meantime from Novikov. For this reason, we attribute Theorem 2.1 together with Corollary 5.8 to Lusin *and* Novikov. And the dual statement of Theorem 3.1 combined with Theorem 5.6 will be referred to as *the Lusin-Novikov Theorem for Partitions (LNTP).*

6. Prikry's manuscript

As phrased in the Introduction, the statement of Proposition 1.1 simplifies the discussion of the early research triggered by Kuratowski's paper. However, the original formulation in [16, Theorem 2] is different. That theorem is stated for a partition of an arbitrary 2nd category *subset* of a Polish space. As Theorem 3.1 is also stated in that generality (Polish space being replaced by a 2nd countable topological space), it looked, at least on the face of it, like LNTP was the first result which openly subsumed Kuratowski's theorem. Then, we focused on Prikry's manuscript again. The proof there, although originally applied in the case of $X = [0, 1]$, is exceptional for its generality. Furthermore, there is a curious passage before Prikry embarks on his statement and proof. He writes that the result is valid, in a stronger form, for any subset $E \subset [0, 1]$ whose outer measure is positive or if E is of 2nd category. Namely, what he claims, amounts in the case of partitions to the non-existence of the separation by measurable sets or by sets having the Baire property in the sense of the LNTP. However, as he writes it, he restraints himself from going into this more general situation "in order to avoid extra explanation".

In conclusion, Prikry actually proved (compare Corollaries 4.1 and 4.2 in [11])

Proposition 6.1. *Let Y be a 2nd countable topological space equipped with a finite regular Borel measure μ, and let $\{A_\gamma : \gamma \in \Gamma\}$ be a point finite family of subsets of Y with union $E = \bigcup_{\gamma\in\Gamma} A_\gamma$. Suppose that for any $\gamma \in \Gamma$, $\mu(A_\gamma) = 0$ (resp., A_γ is 1st category) and $\mu_e(E) > 0$ (resp., E is 2nd category). Then there exists $\Theta \subset \Gamma$ such that $\bigcup_{\gamma\in\Theta} A_\gamma$ is not measurable (resp., does not have the Baire property).*

We recall that $\{A_\gamma : \gamma \in \Gamma\}$ is said to be *point finite*, if each point x in $E = \bigcup_{\gamma\in\Gamma} A_\gamma$ belongs to at most finite number of A_γ's. Prikry's proof of Proposition 6.1 remained in the folklore; it can now be found in [17, Proof of Lemma 3.2]. Note that although Lemma 3.2 in

[2] It is now known that the result is impossible in ZFC. Indeed, it is consistent [21] that there exists a so called Q-set E which is uncountable, and whose every subset is a relative G_δ set. In particular, every subset of E is the trace of a Borel subset of real line.

[17] is stated in a weaker form, its proof gives Propositon 6.1. An additional hint is that it is better to carry on the proof with respect to the whole space Y instead of simplifying the notation by taking $\beta = \Gamma$.

With the assumptions of the above Proposition 6.1, here is the stronger statement claimed by Prikry without proof.

Theorem 6.2. *There exists $\Theta \subset \Gamma$ such that there is no m-measurable (resp., having the Baire property) subset C of Y such that $C \cap E = \bigcup_{\gamma\in\Theta} A_\gamma$.*

We will prove it by reducing Theorem 6.2. to Propositon 6.1.

1. The measure case.

For the purpose of the proof, it will be convenient to denote the space in the theorem by X (keeping its measure to be m). With this setting, denote by $\mathcal{B}$ the σ-algebra of Borel subsets of X, by $\mathcal{B}_E$ its restriction to E (i.e., the Borel σ-algebra of the subspace E of X), and let μ be defined as the restriction of m_e, the outer measure of m, to $\mathcal{B}_E$, that is, $\mu(A) = m_e(A)$ for $A \in \mathcal{B}_E$. Then μ is a regular Borel measure on E and $\mu(A_\gamma) = 0$ for each $\gamma \in \Gamma$. Furthermore, define $\mathcal{N}^\mu = \{Z \subset E : \mu_e(Z) = 0\}$ and $\mathcal{N}^m = \{Z \subset X : m_e(Z) = 0\}$.

By applying Proposition 6.1 with $Y = E$ and the measure μ, we infer the existence of $\Theta \subset \Gamma$ such that $\bigcup_{\gamma\in\Theta} A_\gamma$ is not μ-measurable, i.e., is not a member of the μ-completion of $\mathcal{B}_E$. Thus, we have $\bigcup_{\gamma\in\Theta} A_\gamma \notin \mathcal{B}_E \vartriangle \mathcal{N}^\mu$, where $\vartriangle$ stands for the symmetric difference. Now, denote by $\mathcal{M}$ the family of m-measurable sets in X and observe that

$$\mathcal{M} \cap E = (\mathcal{B} \vartriangle \mathcal{N}^m) \cap E = \mathcal{B}_E \vartriangle \mathcal{N}^\mu.$$

Hence, no m-measurable subset C of X such that $C \cap E = \bigcup_{\gamma\in\Theta} A_\gamma$ may exist. This ends the proof in the 'measure case'.

2. *The Baire case.*

As above, we denote the space in the statement of Theorem 6.2 by X. Let Q be a countable dense subset of X and set $E_1 = E \cup Q$. As E_1 is dense in X, by [15, §10.IV, Theorem 2], the sets A_γ are 1st category relative to E_1. By Proposition 6.1 with $Y = E_1$, we get the existence of $\Theta \subset \Gamma$ such that $A = \bigcup_{\gamma\in\Theta} A_\gamma$ does not have the Baire property relative to E_1. Suppose that there exists C with the Baire property relative to X such that $C \cap E = A$. Then $C \cap E_1 = A \cup Q_2$, where $Q_2 \subset Q$. As E_1 is dense in X, by [15, §11.V, Theorem 2], $C \cap E_1$ has the Baire property relative to E_1, and so $A \cup Q_2$ has it. It follows that A has the Baire property relative to E_1, in contradiction with the choice of A. This ends the proof in the 'Baire case'.

Remark 6.3. One can easily check that, by essentially repeating their proofs, also Theorems 3.3 and 5.9 admit the following strengthening to the point-finite case.

Let Y be a 2nd countable topological (resp., Polish) space. Suppose that $\{A_\gamma : \gamma \in \Gamma\}$ is a point-finite family of universally $\mathcal{M}$-null (resp., always 1st category) subsets of Y. If the union $E = \bigcup_{\gamma\in\Gamma} A_\gamma$ is not an $\mathcal{M}$-null (resp., always 1st category) subset of Y, then there exists $\Theta \subset \Gamma$ such that there is no universally $\mathcal{M}$-measurable (resp., having the Baire property in the restricted sense) subset C such that $C \cap E = \bigcup_{\gamma\in\Theta} A_\gamma$.

7. Appendix

§1. Kunugi's statement in his proof (Section 4), in which he refers to a *result* of Montgomery, is slightly misleading. What he really means is that by applying a similar *method of proof* as the one used by Montgomery in [22], one can get

Proposition 7.1. *Let $Y = (Y, \rho)$ be a metric space and $\varepsilon > 0$. There exist sets $Y^n(\varepsilon), n \in \mathbb{N}$, such that $Y = \bigcup_n Y^n(\varepsilon)$ and $Y^n(\varepsilon)$ decomposes into a transfinite sequence $Y^n_\xi(\varepsilon)$ of G_ρ-sets in such a way that $\rho(Y^n_\xi(\varepsilon), Y^n_{\xi'}(\varepsilon)) > \frac{1}{n}$ for each $\xi \neq \xi'$ and the diameter $d(Y^n_\xi(\varepsilon)) < \varepsilon$ for each ξ.*

Proof. Let $\{B_\gamma : \gamma \in \Gamma\}$ be a base of open sets in Y consisting of sets whose diameter $d(B_\gamma) < \varepsilon$. Define $H_\gamma = B_\gamma \setminus \bigcup_{\eta<\gamma} B_\eta$ and set

$$Y^n_\gamma(\varepsilon) = \{x \in H_\gamma : \rho(x, Y \setminus B_\gamma) > \frac{1}{n}\} = H_\gamma \cap \{x \in B_\gamma : \rho(x, Y \setminus B_\gamma\} > \frac{1}{n}\}.$$

As the latter set is an intersection of a G_ρ-set and an open set, $Y^n_\gamma(\varepsilon)$ is a G_ρ-set.

Some of these sets $Y^n_\gamma(\varepsilon)$ may be empty. Removing all the empty sets, we will get a new index set. Let us denote its indices by ξ. It is easily seen that, for each $\xi \neq \xi'$, $\rho(Y^n_\xi(\varepsilon), Y^n_{\xi'}(\varepsilon)) > \frac{1}{n}$. As $Y^n_\xi(\varepsilon) \subset H_\xi \subset B_\xi$, its diameter is less than ε. If $x \in Y$, then $x \in H_\xi$ for some ξ. As $x \in B_\xi$, an open set, $\rho(x, Y \setminus B_\xi) > 0$. Choose n_0 so that $\frac{1}{n_0} < \rho(x, Y \setminus B_\xi)$. Then $x \in Y^{n_0}_\xi(\varepsilon) \subset Y^{n_0}(\varepsilon)$. Hence $Y = \bigcup_n Y^n(\varepsilon)$. □

§2. After the proof of the existence of two subsets of E that cannot be separated by measurable sets, Lusin gave the following conclusion without any further explanation:

In the end, we have two disjoint subsets E_1 and E_2 of E contained in a perfect set Q of positive measure such that each one of them has outer measure equal to the measure of Q.

Remark 7.2. For a stronger result of this type, in whose proof the above conclusion is used, see [27, Théorème II]. See also [8].

Let us show that Lusin's statement is indeed true (in our more general setting, we will have to assume that the measure vanishes on points). We need the following

Lemma 7.3. *Let Q be a measurable set contained in an envelope of A. Then Q is an envelope of $Q \cap A$.*

Proof. Denote by $\tilde{A}$ an envelope of A. Let G be a measurable set contained in $Q \setminus (Q \cap A)$. Then $G \subset \tilde{A} \setminus A$ and so, by Lemma 5.1, $m(G) = 0$. The same Lemma implies that Q is an envelope of $Q \cap A$. □

Now, let A and B be two disjoint subsets of E that cannot be separated by a measurable set. Let $S = \tilde{A} \cap \tilde{B}$. Then S is of positive measure, because $\tilde{A} \cap B$ already is, as otherwise A and B could be separated. At this point, we assume additionally that (Y is T_1 and) m *vanishes on points*. As $m(S) > 0$, we can find a closed subset of S of positive measure by regularity of m and remove points that are not its condensation points (cf. [5, 1.7.11]) to get a perfect set Q. Then, by Lemma 7.3, $m(Q) = m_e(Q \cap F_1) = m_e(Q \cap F_2)$. Set $E_1 = Q \cap F_1$ and $E_2 = Q \cap F_2$.

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