

ON TWO THEOREMS OF SIERPIŃSKI

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ABSTRACT. A theorem of Sierpiński says that every infinite set Q of reals contains an infinite number of disjoint subsets whose outer Lebesgue measure is the same as that of Q . He also has a similar theorem involving Baire property. We give a general theorem of this type and its corollaries, strengthening classical results.

1. MOTIVATION AND DEFINITIONS

The main result of this note is motivated by two theorems of Sierpiński which, in turn, are connected with the results and techniques due to Lusin and Novikov [10]. The first theorem ([15, Theorem II]) is

Theorem 1.1. *Let Q be an infinite subset of \mathbb{R} . Then Q contains an infinite number of disjoint subsets, each of which has the same outer Lebesgue measure as Q .*

The second (see Théorème 1 and a remark following it in [14, Supplément]) is

Theorem 1.2. *Let Q be a subset of an interval J . If Q is everywhere 2nd category in J , then it is the union of an infinite number of disjoint subsets, each of which is everywhere 2nd category in J .*

We recall that Q is said to be everywhere 2nd category in J if Q is 2nd category at each point $x \in J$. The latter means that for each neighborhood V of x , the intersection $V \cap Q$ is 2nd category.

When searching for a common abstraction of those results, one needs to find a proper notion of the ‘size’ of a subset E in Q , a notion that would cover the measure case as well as the Baire category case.

Our basic framework is a *triple* $(X, \mathcal{A}, \mathcal{J})$, where X is a set, \mathcal{A} a σ -algebra of its subsets (which are called *measurable*), and \mathcal{J} a σ -ideal of its subsets (which are called *negligible*) such that $\mathcal{J} \subset \mathcal{A}$. We will use the following definition that goes back at least to [3] (in that paper the subset E was qualified as *being of the same size as Q*).

Let E and Q be subsets of X , with $E \subset Q$. We say that E is a *full subset of Q* if, for each $B \in \mathcal{A}$ containing E , the set B must contain Q modulo \mathcal{J} , that is, one has $Q \setminus B \in \mathcal{J}$. In a situation in which one would like to specify the σ -ideal, we may also write that E is an \mathcal{J} -full subset of Q . One easily shows

Proposition 1.3. *The following are equivalent.*

- (a) E is a full subset of Q .
- (b) For each $A \in \mathcal{A}$, if $A \cap Q \notin \mathcal{J}$, then $A \cap E \notin \mathcal{J}$.

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Following [16], let us say that a subset E of Q is $(\mathcal{A}, \mathcal{J})$ -completely non-measurable in Q if the following condition is satisfied: for each measurable set A , if $A \cap Q \notin \mathcal{J}$, then $A \cap E \notin \mathcal{J}$ and $A \cap (Q \setminus E) \notin \mathcal{J}$.

In view of the condition (b) in the Proposition above, we have

Proposition 1.4. *Let $E \subset Q$. Then E is $(\mathcal{A}, \mathcal{J})$ -completely non-measurable in Q iff E and $Q \setminus E$ are full subsets of Q .*

Consequently, if $\{E_1, E_2\}$ is a decomposition of a set into disjoint full subsets, then it is its decomposition into $(\mathcal{A}, \mathcal{J})$ -completely non-measurable subsets.

In what follows we simply write *completely non-measurable*, since there will not be any ambiguity about \mathcal{A} and \mathcal{J} .

2. RESULT

Let F be a subset of X . We say that a set \tilde{F} containing F is a *measurable envelope* of F if $\tilde{F} \in \mathcal{A}$ and, for each measurable A , the containment $A \subset \tilde{F} \setminus F$ implies $A \in \mathcal{J}$. Below ‘envelope’ will mean ‘measurable envelope’. It is easily seen that a measurable set Q is an envelope of E iff E is full in Q . Further, we have

Lemma 2.1. *Let Q be a measurable set contained in \tilde{F} . Then Q is an envelope of $Q \cap F$. In particular, $Q \cap F$ is full in Q .*

Indeed, let A be a measurable set contained in $Q \setminus (Q \cap F)$. Then $A \subset \tilde{F} \setminus F$ and so $A \in \mathcal{J}$. Hence Q is an envelope of $Q \cap F$.

Let a triple $(X, \mathcal{A}, \mathcal{J})$, as defined in Section 1, be given. *Throughout this Section we impose the following two assumptions on $(X, \mathcal{A}, \mathcal{J})$.*

- The family $\mathcal{A} \setminus \mathcal{J}$ satisfies the countable chain condition CCC, that is, every disjoint family of non-negligible measurable sets is countable. It is well-known and easily shown that under CCC each subset of X admits an envelope.
- X has the *non-separation property with respect to \mathcal{A}* . Namely, if $E \notin \mathcal{J}$, then there exist disjoint subsets of E which cannot be separated by sets from \mathcal{A} (in particular, the singletons must belong to \mathcal{J}).

Lemma 2.2. *If $E \notin \mathcal{J}$, then there exist a measurable set $Q \notin \mathcal{J}$ and disjoint sets E_1, E_2 contained in $E \cap Q$ such that E_1 and E_2 are full subsets of Q .*

Proof. Let F and G be disjoint subsets of E that cannot be separated by measurable sets. Denote by \tilde{F} and \tilde{G} their respective envelopes. Put $Q = \tilde{F} \cap \tilde{G}$. As F and G are not separated, $\tilde{F} \cap \tilde{G} \notin \mathcal{J}$, since otherwise $\tilde{F} \setminus (\tilde{F} \cap \tilde{G})$ would be a measurable set separating F and G . Hence $Q = \tilde{F} \cap \tilde{G}$ is not negligible. By Lemma 2.1, Q is an envelope of $F \cap Q$ and also of $G \cap Q$. Thus, $E_1 = F \cap Q$ and $E_2 = G \cap Q$ are full in Q . \square

Remark 2.3. Envelopes are also instrumental in characterizing fullness. Notably, the following Proposition holds: *Let $E \subset Q \subset X$ and assume that Q admits an envelope. Then E is a full subset of Q iff any envelope of Q is an envelope of E .*

Here is our main result.

Theorem 2.4. *Each infinite subset E of X decomposes into two disjoint full subsets. Consequently, it can be written as the infinite disjoint union of its full (equivalently – completely non-measurable) subsets.*

Proof. We first show the decomposition into two subsets. We may assume that $E \notin \mathcal{J}$.

Let us say that a set P has the equal sizes property relative to E whenever P is a non-negligible measurable set such that there exist disjoint sets E_1, E_2 contained in $E \cap P$, each one full in P .

Let \mathcal{F} be a maximal disjoint family of sets $\{Q_\xi\}$ having that property. It is not empty by Lemma 2.2. By the countable chain condition, $\mathcal{F} = Q_1, Q_2, \dots$. Observe first that

$$(*) \quad E^* = E \setminus \bigcup_{n=1}^{\infty} Q_n \in \mathcal{J}.$$

Indeed, otherwise one could apply Lemma 2.2 to E^* and find a corresponding set Q^* . That is, Q^* belongs to $\mathcal{A} \setminus \mathcal{J}$ and there exist disjoint subsets E_1 and E_2 of $E^* \cap Q^*$ which are both full in Q^* . We thus have E_1 and E_2 contained in A , where $A = Q^* \setminus \bigcup_n Q_n$. As $A \subset Q^*$, the sets E_1, E_2 are full in A . Also, $A \notin \mathcal{J}$ since it contains E_1 which is not in \mathcal{J} . To sum up, A is disjoint with all Q_n 's and has the equal sizes property relative to E . This contradicts the fact that \mathcal{F} is maximal.

Now, for each $n \in \mathbb{N}$, let C_n and D_n be disjoint subsets of $E \cap Q_n$ that, moreover, are full subsets of Q_n . We claim that $C = \bigcup_n C_n$ and $D = \bigcup_n D_n$ are full subsets of E . Indeed, let C be contained in a measurable set B . Then $C_n \subset B$ and so $Q_n \setminus B \in \mathcal{J}$ for each $n \in \mathbb{N}$. This implies that $(\bigcup Q_n) \setminus B \in \mathcal{J}$ and finally, by (*), that $E \setminus B \in \mathcal{J}$. For $D \subset B$, the argument is similar. If D is not the complement of C in E , we can replace it by that complement in order to get the needed decomposition.

By what we have just shown, E is the union of two full subsets, E_1 and its complement E'_1 in E . Next, by the same argument, decompose E_1 into E_2 and E'_2 . It can readily be checked that E_2 , as a full subset of a full subset of E , is full in E . Proceed inductively. \square

Remark 2.5. The proof above is a modification of Sierpiński's argument in the proof of Théorème I in [15]. It is Lemma 2.2 that makes the passage to the abstract case possible. For the Lebesgue measure, Lusin stated in [10] a fact stronger than Lemma 2.2 (see [4, Appendix]) and it was this stronger fact that Sierpiński used in his proof of Theorem 1.1.

Consider triples $(X, \mathcal{A}_n, \mathcal{J}_n)$, $n \in \mathbb{N}$, where \mathcal{A}_n are σ -algebras and $\mathcal{J}_n \subset \mathcal{A}_n$ are σ -ideals in X , together with $(X, \mathcal{A}^\infty, \mathcal{N})$, where the σ -algebra $\mathcal{A}^\infty = \bigcap_n \mathcal{A}_n$ and the σ -ideal $\mathcal{N} = \bigcap_n \mathcal{J}_n$. Assume that the triples $(X, \mathcal{A}_n, \mathcal{J}_n)$ satisfy the assumptions specified by the bullets above.

Proposition 2.6. *The family $\mathcal{A}^\infty \setminus \mathcal{N}$ satisfies CCC and X has the non-separation property with respect to \mathcal{A}^∞ .*

Proof. Let \mathcal{E} be a family of disjoint sets in \mathcal{A}^∞ that do not belong to \mathcal{N} . Then, $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$, where $\mathcal{E}_n \subset \{A \in \mathcal{A}^\infty : A \notin \mathcal{J}_n\} \subset \{A \in \mathcal{A}_n : A \notin \mathcal{J}_n\}$. Hence each \mathcal{E}_n is countable, and so \mathcal{E} is countable as well.

Suppose E is a subset of X such that $E \notin \mathcal{N}$. Then, $E \notin \mathcal{J}_n$ for some $n \in \mathbb{N}$. Consequently, there exist two disjoint subsets of E that cannot be separated by sets from \mathcal{A}_n . A fortiori, they cannot be separated by sets from \mathcal{A}^∞ . \square

Corollary 2.7. *Each infinite subset E of X decomposes into two disjoint \mathcal{N} -full subsets. Consequently, it can be written as the infinite disjoint union of its subsets, each of which is \mathcal{N} -full (equivalently – completely non-measurable with respect to \mathcal{A}^∞).*

Let $E = \bigcup_{i=1}^\infty E_i$ be the decomposition obtained for the set E in the above Corollary.

Proposition 2.8. *Assume that, for each $n \in \mathbb{N}$ and each $A \in \mathcal{A}_n$, there exists $B \in \mathcal{A}^\infty$ such that $A \subset B$ and $B \setminus A \in \mathcal{J}_n$. Then each E_i in the decomposition is \mathcal{J}_n -full (equivalently – completely non-measurable with respect to \mathcal{A}_n) simultaneously for all $n \in \mathbb{N}$.*

Proof. Let C be a subset of Q that is full with respect to \mathcal{N} and \mathcal{A}^∞ . Fix n and take $A \in \mathcal{A}_n$ containing C . Find $B \in \mathcal{A}^\infty$ containing A and such that $B \setminus A \in \mathcal{J}_n$. As $B \supset C$, one has $Q \setminus B \in \mathcal{N} \subset \mathcal{J}_n$. Consequently, $Q \setminus A \in \mathcal{J}_n$ and C is \mathcal{J}_n -full in Q . \square

Remark 2.9. Let \mathcal{B} be a σ -algebra, and \mathcal{J} a σ -ideal in X . Then \mathcal{J} is said to have a *base* in \mathcal{B} if, for each $J \in \mathcal{J}$, there exists $K \in \mathcal{J}$ containing J and belonging to \mathcal{B} .

Suppose that $\mathcal{A}_n = \mathcal{B} \Delta \mathcal{J}_n$, where Δ stands for the symmetric difference. It is easy to see that, if \mathcal{J}_n has a base in \mathcal{B} , then, for each $A \in \mathcal{A}_n$, there exists $B \in \mathcal{B}$ containing A and such that $B \setminus A \in \mathcal{J}_n$. In particular, the assumptions in the above Proposition are satisfied.

3. APPLICATIONS

§1. μ -measurability. Let μ be a complete σ -finite (positive, countably additive) measure on a set X . Let \mathcal{A} be the σ -algebra of μ -measurable sets and $\mathcal{J} = \mathcal{J}$ its σ -ideal of μ -zero sets. Write μ^* the outer measure of μ , and let E and Q be subsets of X with $E \subset Q$. We note that if $\mu^*(Q) < \infty$, then E is \mathcal{J} -full subset of Q iff $\mu^*(E) = \mu^*(Q)$.

Corollary 3.1. *Let X be a 2nd countable topological T_1 -space, and μ the completion of a regular Borel σ -finite measure vanishing on points. Then every infinite subset E of X can be decomposed into an infinite disjoint union of its \mathcal{J} -full (equivalently – completely non-measurable) subsets. In particular, each of these subsets has the same outer measure as E .*

Proof. It is well known that $\mathcal{A} \setminus \mathcal{J}$ has CCC. The non-separation property of X with respect to \mathcal{A} follows from the Lusin-Novikov Theorem [4, Corollary 5.8]. \square

§2. Baire property. Let us recall that a subset E of a topological space X has the *Baire property* (or is *BP-measurable*) if it can be written in the form $E = (O \setminus P) \cup Q$, where the set O is open, while P and Q are 1st category. Let $\mathcal{A} = \mathcal{B} \Delta \mathcal{J}$ be the σ -algebra of sets having the Baire property in X , where \mathcal{B} denotes the σ -algebra of Borel sets and $\mathcal{J} = \mathcal{K}$ is the σ -ideal of 1st category (meager) sets. If X is 2nd countable, then $\mathcal{A} \setminus \mathcal{K}$ has CCC ([9, §24.I, Theorem 3]). If X is a 2nd countable T_1 -space without isolated points, then X has the non-separation property with respect to \mathcal{A} by the Lusin-Novikov Theorem (see [4, Corollary 3.2]). Let us add that the existence of envelopes does not depend on CCC. Every subset $F \subset X$ has an F_σ -envelope \tilde{F} (Szpilrajn-Marczewski's theorem [9, §11.IV, Corollary 1]).

In the ‘Baire property case’ completely non-measurable subsets of a set Q were called *completely non-Baire subsets of Q* in [2]. Following this terminology, we have

Corollary 3.2. *Let X be a 2nd countable topological T_1 -space without isolated points. Then every infinite subset of X can be decomposed into an infinite disjoint union of its \mathcal{K} -full (equivalently – completely non-Baire) subsets.*

Remark 3.3. If X is an uncountable Polish space, a generalization of the above Corollary for partitions into 1st category sets holds ([2, Theorem 6.8]).

We recall that a topological space is called a *Baire space* if its nonempty open subsets are 2nd category.

Proposition 3.4. *If $E \subset X$ is 2nd category at each point $x \in X$, then E is full in X . Conversely, if X is a Baire space and E is full in X , then E is 2nd category at each point $x \in X$.*

Proof. For a subset A of X , denote by $D(A)$ the set of all points in which A is 2nd category. Let B be a set having the Baire property and containing E . Then, by [9, §11. IV.5], $D(B) \setminus B$ is of 1st category. Since $D(B) \supset D(E) = X$, we have that $X \setminus B$ is 1st category.

To see the converse, observe that, by the condition (b) of Proposition 1.3, E is full in a set $Q \subset X$ means that for each $A \in \mathcal{A}$, if $A \cap Q \notin \mathcal{J}$, then $A \cap E \notin \mathcal{J}$. It is not difficult to show, that it is sufficient to take open sets in the above condition. Thus, if $Q = X$, we have that E is full in X means that for each open set A which is of 2nd category, $E \cap A$ is 2nd category. Let V be an open neighborhood of $x \in X$. As X is assumed to be a Baire space, V is 2nd category, and so $V \cap E$ is 2nd category as well. \square

Theorem 1.2, a solution to a problem posed by Kuratowski [8, Problème 21], can be generalized as follows.

Corollary 3.5. *Let X be a 2nd countable topological Baire T_1 -space without isolated points and $E \subset X$. If E is everywhere 2nd category in X , then it can be decomposed into the union of an infinite number of disjoint subsets, each of which is everywhere 2nd category in X .*

Proof. If E is everywhere 2nd category in X , then it is full there. Applying Corollary 3.2, E can be decomposed into its two full subsets E_1 and E_2 , which are therefore full in X . Hence they are both everywhere 2nd category in X . Proceed inductively. \square

§3. Other examples.

In view of Proposition 2.8 and the remark following it, we have

Corollary 3.6. *With the assumptions of Corollaries 3.1 and 3.2 combined, each infinite subset of X admits a countable decomposition into subsets that are full (or completely non-measurable) in terms of measure and Baire property simultaneously.*

Under additional set-theoretical assumptions about X , decompositions into more than a countable number of subsets are possible. Actually, the main result of [15], the already mentioned Théorème I, is concerned with an uncountable case. We now give a general result of this type.

Let \mathcal{A} be a σ -algebra of subsets of a set X containing all singletons. Define $\mathcal{J} = \mathcal{J}(\mathcal{A})$ as the σ -ideal of those sets in \mathcal{A} which are hereditarily in \mathcal{A} , i.e., if A belongs to \mathcal{A} together with each subset C of A , then $A \in \mathcal{J}$. Taking into account [3, Theorem 2], we have

Corollary 3.7. *Suppose the cardinality of X is less than the first weakly inaccessible cardinal and $\mathcal{A} \setminus \mathcal{J}$ satisfies CCC. Then each uncountable subset Y of X can be decomposed into \aleph_1 disjoint full (or completely non-measurable) subsets.*

Further, we were informed by the referee of this paper that, in a separable metric space, under the non-existence of quasi-measurable cardinals less or equal 2^{\aleph_0} , for a CCC σ -ideal \mathcal{J} with a Borel base every point-finite family $\mathcal{A} \subset \mathcal{J}$ can be decomposed into \aleph_1 subfamilies, the union of each of them being full in $\bigcup \mathcal{A}$ relative to \mathcal{J} [12, Theorem 3.6]. Also, under some cardinal coefficients, any $cf(2^{\aleph_0})$ -point family \mathcal{A} of subsets of a σ -ideal \mathcal{J} can be decomposed into 2^{\aleph_0} pairwise disjoint subfamilies, the union of each of them being full in $\bigcup \mathcal{A}$ relative to \mathcal{J} ([12, Corollary 2.1].

4. APPENDIX

4.1. Historical comments. (1) The title of the present note refers to *two theorems of Sierpiński* for the simple reason that these theorems were first published by Sierpiński. However, it ought to be mentioned that in [14, Supplément] Sierpiński considers his Théorème 1 as an easy consequence of a ‘method of Lusin’ and in [9, §10 (p. 87)] the result reappears as ‘Lusin Theorem’ with a sole reference to the Supplément. Both attributions strike us as being not correct. Admittedly, it seems possible that writing his comments in the Supplément Sierpiński could have relayed on a letter from Lusin quoted there and did not yet know [10]. However, in [15] Sierpiński refers directly to [10], in which paper Lusin himself admits that the method of proof is due to Novikov; and still only the name of Lusin is mentioned in [15]. Similarly, writing much later, Kuratowski must have known [10] and its reviews [1], [13], in which the proofs of [10] are referred to as Novikov’s. Yet, he keeps quoting only the Supplément and calls the result ‘Lusin Theorem’. What would be the reason for this silence about Novikov, we do not know.

(2) Ashutosh Kumar stated in his thesis [5, Theorem 1] the decomposition of a subset of \mathbb{R}^n into two disjoint full subsets in the Baire property case and the Lebesgue measure case. The Baire part of that statement is a new result corresponding to our Corollary 3.1. Kumar attributes the whole theorem to Lusin quoting [10], despite the fact that no such statement is present in [10] and despite the fact that any result quoted from [10] should be attributed to Lusin and Novikov. He repeats the misquotation in [6] and then, together with Shelah, in [7]. We are also aware of the work by Marcin Michalski (although no preprint is available yet), who in his talk [11] gives a general definition of an \mathcal{J} -full subset and announces a proof of the Lebesgue measure and the Baire property statements in \mathbb{R} by using a modification of the technique of [10], still attributing the result to Lusin.

4.2. Kumar’s proof. Consider a triple $(X, \mathcal{A}, \mathcal{J})$ introduced in Section 1 and assume that \mathcal{A} -measurable envelopes exist in the space X . Then, the following proposition holds.

Proposition 4.1. *For a triple $(X, \mathcal{A}, \mathcal{J})$ and a subset Y of X , the following conditions are equivalent.*

- (a) *Y contains $Z \notin \mathcal{J}$ such that $Y \setminus Z$ is full in Y .*
- (b) *Y contains $Z \notin \mathcal{J}$ such that Z and $Y \setminus Z$ cannot be separated by an \mathcal{A} -measurable set.*
- (c) *Y contains disjoint subsets F and G that cannot be separated by an \mathcal{A} -measurable set.*

Proof. The implications from (a) to (b) and from (b) to (c) being evident, only (c) implies (a) has to be shown.

Assume (c), and define $Q = \tilde{F} \cap \tilde{G}$. By applying Lemma 2.1 (with $Y = E$), we have $Q \cap F$ and $Q \cap G$, subsets of $Y \cap Q$, which are full in Q . Then (a) holds with $Z = Q \cap F$.

Indeed, we note that $Y \cap (Q \setminus F)$, containing $Q \cap G$ that is full in Q , must be full in $Q \cap Y$. Obviously $Y \setminus Q$ is full in itself, and so $(Y \setminus Q) \cup [Y \cap (Q \setminus F)]$ is full in $(Y \setminus Q) \cup (Y \cap Q)$. Hence $Y \setminus (Q \cap F)$ is full in Y . \square

Kumar's proof of his Theorem 1 in [5] proceeds first with the measure case. He begins by remarking that one can reduce to consider the Lebesgue measure on $[0, 1]$ and says that it is enough to show that every non-negligible subset $Y \subset [0, 1]$ satisfies the condition (a) above. This is so, because then, one can find the needed decomposition of Y into two full subsets. He defines the needed full subset E without actually showing that the so defined E and its complement are indeed full in Y . In the next step the proof becomes indirect. By denying the needed condition, he concludes that every two disjoint subsets of Y would be separated by a measurable set (in his phrasing the restriction of the outer Lebesgue measure to Y is countably additive on the power set of Y). At this point the proof stops being classical. By a use of forcing, it is shown that the above conclusion about the outer Lebesgue measure on Y leads to a contradiction.

The proof of the Baire case is similar (although the definition of E is not stated and so the proof that it has required properties is not given either). But assuming that E can be well defined, which Kumar silently does, by analogy with the measure case one can find a 2nd category subset Y of $[0, 1]$ such that for every 2nd category subset E of Y its complement $Y \setminus E$ is not full in Y . Then, an appropriately modified use of forcing gives a contradiction.

Here are our comments.

(1) The measure case. It is the contradiction proved by forcing that is actually proven classically in the the quoted note of Lusin [10].

(2) Kumar's reasoning showing the existence of a subset Y whose every two disjoint subsets can be separated by a Lebesgue measurable set remains the same in the general framework of \mathcal{A} -measurability. It shows that, if for each non-negligible subset Z of Y one has $Y \setminus Z$ which is not full in Y , then every two disjoint subsets of Y can be separated by an \mathcal{A} -measurable set. Thus, also in the Baire case the contradiction gotten by forcing is a consequence of a result in [10]

(3) The (omitted by Kumar) argument showing that E and its complement are full in Y can be adapted to the general \mathcal{A} -measurable case. But there is no point giving details, since it follows from Proposition 4.1 that our approach and Kumar's are equivalent.

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