MEASURE, CATEGORY AND CONVERGENT SERIES

Abstract

The analogy between measure and Baire category is first illustrated by a theorem of Steinhaus and its ‘dual’, a theorem of Piccard. These two theorems are then applied to provide a dual criterion for the unconditional convergence of a series in terms of the size of its convergent sub-series. As a further application, after a preparatory section concerning essential separability of measurable functions, a dual result about Haar measurable and \( BP \)-measurable additive maps on the Cantor group is established. The analogy is stressed not only by the results themselves, but also by the proofs provided. Finally, the universal measurability and universal \( BP \)-measurability are considered. Two classical theorems are examined. How far the analogy extends at the ‘universal measurability’ level seems to be an open question.

1 Introduction

There is a long line of research motivated by the analogy between measure and (Baire) category, see e.g. Oxtoby’s survey [28]. In particular, the analogy may concern ‘dual’ statements involving \( BP \)-measurable sets and measurable sets. In those statements one would like 1st category sets to correspond to measure zero sets, and 2nd category sets (those that are not 1st category) to sets of positive measure. We recall that a subset of a topological space is 1st category if it is a countable union of nowhere dense sets. An example of the duality in question is provided by the following two classical results.

First is the Steinhaus Theorem (going back to [32]; in the group setting, to [35]).
Theorem 1.1. Let $G$ be a locally compact group with (left invariant, say) Haar measure $\chi$ and $A \subset G$. If $A$ is of positive measure, then $AA^{-1}$ is a neighborhood of unit in $G$.

Here is an elementary proof due to Kestelman [22] (who still worked in $\mathbb{R}^n$), later rediscovered by Stromberg in [33].

Proof. By the inner regularity of $\chi$, one can assume that $A$ is compact. By the outer regularity of $\chi$, one finds an open set $U$ containing $A$ such that $\chi(U \setminus A) < \frac{1}{2}\chi(A)$, and then a neighborhood $V$ of unit such that $VA \subset U$. Given $z \in V$, $zA \subset VA \subset U$ and so

$$\chi(U \setminus zA) = \chi(U) - \chi(zA) = \chi(U) - \chi(A) < \frac{1}{2}\chi(A).$$

Hence $\chi(U \setminus (A \cap zA)) \leq \chi(U \setminus A) + \chi(U \setminus zA) < \chi(A) < \chi(U)$. Consequently, $\chi(A \cap zA) \not= 0$ and $A \cap zA \not= \emptyset$. Therefore there exists $y \in A$ such that $y = zx$ with $x \in A$, whence $yx^{-1} = z$ and $z \in AA^{-1}$. As $z$ was an arbitrary member of $V$, this means that $V \subset AA^{-1}$. 

The dual statement is a theorem of Piccard [31], mostly known as the Pettis Lemma (because of [30]). Kelley has it as an exercise ([21], Chapter 6, Exercise P(b)) and calls the Banach-Kuratowski-Pettis Theorem.

Recall that a subset $A$ in a topological space $Y$ is said (to have the Baire Property or) to be $BP$-measurable if it can be written as a symmetric difference $A = O \triangle I$, where $O$ is open and $I$ is 1st category (in $Y$).

Theorem 1.2. Let $G$ be a topological group and $A$ a $BP$-measurable subset of $G$. If $A$ is of second category, then $AA^{-1}$ is a neighborhood of unit in $G$.

The following proof, due essentially to Bourbaki (compare the proof of [5, p. 69, Lemme 9], shows the duality we are after.

Proof. As $A$ is 2nd category in $G$, so is $G$. By the Banach category theorem [28, Th. 16.1], every nonempty open set is 2nd category in $G$. In particular, if $A = O \triangle I$, then $O$ is 2nd category and therefore not empty. It suffices to show that $OO^{-1} \subset AA^{-1}$. Let $x \in OO^{-1}$. Then the open set $O \cap xO$ is not empty and therefore 2nd category. Define

$$Z = (O \cap xO) \setminus (A \cap xA).$$

Since $Z \subset (O \setminus A) \cup x(O \setminus A)$, $Z$ is 1st category. Consequently, $A \cap xA \not= \emptyset$. This means that $x \in AA^{-1}$ and $OO^{-1} \subset AA^{-1}$. 

These two theorems will provide a unifying tool in what follows. It is our aim to present some results, new and old, that are ‘dual’ in a similar way as the theorems of Steinhaus and Piccard are. Perhaps even more importantly, we provide proofs that display and stress that sort of duality.

2 Series

In what follows $\mathcal{P}(A)$ stands for all subsets of a set $A$, $\mathcal{F}(A)$ for all its finite subsets; $\mathbb{N} = \{1, 2, \ldots\}$, $\mathcal{P} = \mathcal{P}(\mathbb{N})$, $\mathcal{F} = \mathcal{F}(\mathbb{N})$ and $\mathcal{F}_m = \mathcal{F}\{m, m+1, \ldots\}$. By identifying subsets of $\mathbb{N}$ with their characteristic functions, subfamilies of $\mathcal{P}$ become subsets of the Cantor group $K = \{0, 1\}^\mathbb{N}$, i.e., we regard the set of all sequences of 0’s and 1’s as the countable product of the group $\{0, 1\}$ with addition mod. 2. $K$ is a compact metric Abelian group whose addition in terms of $\mathcal{P}$ is the symmetric difference $\Delta$ of subsets in $\mathcal{P}$. Putting masses $1/2$ at the points 0 and 1 of $\{0, 1\}$, the corresponding product measure $\chi$ on $K$ is its Haar probability measure.

Using the identification, the topological properties of subfamilies in $\mathcal{P}$ always refer to the topological properties of the corresponding subsets of the Cantor group.

If $X$ is a topological Abelian group, its topology can be defined by a family of (group) semi-norms. Then, in order to establish convergence of a (filter or) sequence in $X$, it is sufficient to do it with respect to each semi-norm separately. Equivalently, if $X$ is Hausdorff, one can embed $X$ into a product of complete normed groups and one can argue with respect to each coordinate group separately. For a possibility of such an embedding, see e.g. [6, Ch. 2, §1, no 3, Prop. 3], where a full proof is given for the case of a locally convex space, but the argument is general and works in a group setting as well.

From now on, unless stated otherwise, $X$ stands for a Hausdorff topological Abelian group and, if it is normed, then its (group) norm is denoted by $\|\cdot\|$, i.e., $X = (X, \|\cdot\|)$.

In order to set the terminology, recall a few known facts.

Let $X$ be sequentially complete. The following conditions are equivalent for a sequence $(x_n)$ in $X$.

(a) The series $\sum_n x_n$ is unconditionally convergent, i.e., converges for each ordering of its terms.

(b) The series $\sum_{n=1}^{\infty} x_n$ is subseries convergent, i.e., all its partial series are (subseries) convergent.

(c) With the convention $0 \cdot x = 0$ and $1 \cdot x = x$, the series $\sum \varepsilon_n x_n$ converges for all sequences $(\varepsilon_n)$ of 0’s and 1’s.
(d) The series $\sum_n x_n$ satisfies the Cauchy condition:

$$\forall U \exists m \forall F \in \mathcal{F}_m \sum_{n \in F} x_n \in U,$$

where $U$ is a neighborhood of zero in $X$.

(e) If $X$ is normed, then the Cauchy condition can be written as

$$\forall \varepsilon > 0 \exists m \forall F \in \mathcal{F}_m \left\| \sum_{n \in F} x_n \right\| \leq \varepsilon.$$

- For a sequence $x = (x_n)$ in $X$, its set of unconditional convergence is defined as
  $$\mathcal{C}(x) = \mathcal{C}(x_n) = \{ A \in \mathcal{P} : \sum_{n \in A} x_n \text{ is unconditionally convergent} \}.$$

It is identified with

$$\mathcal{C}(x) = \{(\varepsilon_n) \in K : \sum_n \varepsilon_n x_n \text{ is unconditionally convergent}\}.$$

Because of the identification, we use the symbol $\mathcal{C}(x)$ only.

**Proposition 2.1.** Let $X = (X, \|\cdot\|)$ be complete. For every sequence

$$x = (x_n) \subset X,$$

the set $\mathcal{C}(x)$ is an $F_{\sigma\delta}$ subset of $K$.

**Proof.** By using the Cauchy condition (e) above, one has

$$(\varepsilon_j) \in \mathcal{C}(x) \iff \forall k \exists n \forall F \in \mathcal{F}_n : \left\| \sum_{j \in F} \varepsilon_j x_j \right\| \leq \frac{1}{k},$$

hence

$$\mathcal{C}(x) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{F \in \mathcal{F}_n} \left\{ (\varepsilon_j) \in K : \left\| \sum_{j \in F} \varepsilon_j x_j \right\| \leq \frac{1}{k} \right\}.$$

As, for each finite set $F$, the mapping $(\varepsilon_j) \to \left\| \sum_{j \in F} \varepsilon_j x_j \right\|$ from $K$ into $\mathbb{R}$ is continuous, the sets on the right-hand side of the above formula are closed. Consequently, $\mathcal{C}(x)$ is an $F_{\sigma\delta}$ set. \qed
Theorem 2.2. Let $X$ be sequentially complete and $(x_n)$ a sequence in $X$. If $C = C(x_n)$ is either 2nd category or of positive Haar measure, then $\sum_n x_n$ is unconditionally convergent in $X$.

Proof. Let $C$ be 2nd category in $P$. Embed $X$ into a product of complete normed groups. Assume, for a moment, that $X$ is a complete normed group. Then, by Proposition 2.1, $C$ is an $F_{\sigma\delta}$ set in $P$ and so, in particular it is $BP$-measurable. By Theorem 1.2 the family $V = C \triangle C$ is a neighbourhood of zero. Find a finite set $F \in V$, and then $n$ so large that $F \cup H \in V$ with $H = \{n, n + 1, n + 2, \ldots\}$ disjoint with $F$. Then $F \cup H = C_1 \triangle C_2$ with $C_1, C_2$ belonging to $C$. Now $C_1 \setminus (C_1 \cap C_2)$ and $C_2 \setminus (C_1 \cap C_2)$ are clearly in $C$ and so their disjoint union $F \cup H$ is in $C$ as well. In particular, $\sum_{k=n}^{\infty} x_k$ is unconditionally convergent. As this is true for every coordinate group in the product, the original series satisfies the Cauchy condition for summability in $X$ and, since $X$ is sequentially complete, is unconditionally convergent there. If $C$ is of positive Haar measure, the proof is the same applying Theorem 1.1.

Remark 2.3. A family $I \subset P(A)$ is an ideal in $A$, if $B \subset C \in I \implies B \in I$ and $B, C \in I \implies B \cup C \in I$. Suppose $I$ is an ideal in $\mathbb{N}$ containing finite sets. The proof above shows that if $I$ is either 2nd category $BP$-measurable or is of positive Haar measure in $P$, then $I = P$, compare [11].

Remark 2.4. By noticing that $I$ is also a subgroup of $P$, one can apply instead of the Piccard theorem an earlier result of Banach [2, Ch. I, Th. 1].

3 Essential separability

Let $f$ be a function between topological spaces $Y$ and $X$. It has the Baire Property or is $BP$-measurable, if for every open set $V$ in $X$, the set $f^{-1}(V)$ is $BP$-measurable in $Y$. Suppose $\mu$ is a finite (positive) measure on $Y$. Then $f$ is $\mu$-measurable, if for every open set $V$ in $X$, the set $f^{-1}(V)$ is $\mu$-measurable in $Y$.

We will say that $f$ is essentially separably valued in each of the following two (dual) cases. In the $BP$-measurable case: if there exists a 1st category set $A \subset Y$ such that that $f(Y \setminus A)$ is separable. Dually, in the measurable case: if there exists a set $A \subset Y$ of $\mu$-measure zero such that $f(Y \setminus A)$ is separable. A subset in a topological space is called residual if it is the complement of a 1st category set.

Lemma 3.1. Let $X$ be a metric space. If a function $f : Y \to X$ has the property that each open cover of $X$ admits a countable subfamily $B = \{B_1, B_2, \ldots\}$
such that \(\mu(\bigcup_{n=1}^{\infty} f^{-1}(B_n))\) is of full measure (resp., is residual), then \(f\) is essentially separably valued.

**Proof.** For every \(k \in \mathbb{N}\), let us cover \(X\) with \(1/k\) balls and choose countable subfamilies \(B_k = \{B_{k1}, B_{k2}, \ldots\}\) according to the assumption of the lemma. Set \(A_{kn} = f^{-1}(B_{kn}), \ n = 1, 2, \ldots\) Then \(Y_1 = \bigcap_k \bigcup_n A_{kn}\) is of full measure (resp., residual) and its image \(X_1 = f(Y_1)\) is separable. Indeed, one can find a point in \(B \cap X_1\) for each ball \(B \in \bigcup B_k\) and these points form a countable dense subset of \(X_1\).

The following fact can already be deduced from [26], see [18]. Here, its somewhat more modern treatment is adapted from an unpublished manuscript of Prikry, cited as reference number 27 in [23]. We recall that a topological space is said to be 2nd countable if it admits a countable base for open sets.

**Lemma 3.2.** Let \(Y\) be a 2nd countable topological space equipped with a finite regular Borel measure \(\mu\). Let \(\{A_\gamma : \gamma \in \Gamma\}\) be a disjoint family of subsets of \(Y\) so that for any \(\gamma \in \Gamma\), \(\mu(A_\gamma) = 0\) (resp., \(A_\gamma\) is 1st category). If \(\bigcup_{\gamma \in \Gamma} A_\gamma\) has positive outer measure (resp., is 2nd category), then there exists \(\Delta \subset \Gamma\) such that \(\bigcup\{A_\gamma : \gamma \in \Delta\}\) is not \(\mu\)-measurable (resp., BP-measurable).

**Proof.** Well-order \(\Gamma\) and let \(\Theta\) to be the least element in \(\Gamma\) such that, writing \(A = \bigcup_{\gamma < \Theta} A_\gamma\), the outer measure \(\mu^*(A) > 0\) (resp., \(A\) is 2nd category). We may assume that the union \(A\) is \(\mu\)-measurable (resp., BP-measurable), as otherwise the lemma holds. By the definition of \(\Theta\), for \(\beta < \Theta\), \(\mu(\bigcup_{\gamma < \beta} A_\gamma) = 0\) (resp., \(\bigcup_{\gamma < \beta} A_\gamma\) is 1st category).

For any \(\beta < \Theta\), find a \(G_\delta\) set \(P_\beta\) of measure zero containing \(\bigcup_{\gamma < \beta} A_\gamma\) (resp., a residual \(G_\delta\) set \(P_\beta\) disjoint with \(\bigcup_{\gamma < \beta} A_\gamma\)). Define \(E \subset Y \times Y\) by

\[ E = \bigcup_{\gamma < \Theta} (A_\gamma \times P_\gamma) \]

and let \(E_y\) be a horizontal cross-section of \(E\).

Take \(\alpha\) so that \(y \in A_\alpha\) and \(\beta\) such that \(\alpha < \beta < \Theta\). Consider first the measure case. We have \(y \in P_\beta\) and, consequently, \(E_y \supset \bigcup_{\alpha < \beta < \Theta} A_\beta\). Hence, \(\mu(E_y) \geq \mu(A)\), provided \(E_y\) is measurable. In the category case, we have \(y \in \bigcup_{\gamma < \beta < \Theta} A_\gamma \subset Y \setminus P_\beta\). This implies that \(E_y \cap \bigcup_{\alpha < \beta < \Theta} A_\beta = \emptyset\), and \(E_y\) is of 1st category.

Clearly, by the very definition of \(E\), every vertical cross-section \(E^y\) has measure zero (resp., is residual). By the Fubini Theorem (resp., its category analogue, the Kuratowski-Ulam Theorem [28, Chapter 15]), the set \(E\) cannot be measurable (resp., BP-measurable) in \(Y \times Y\) (with its product measure and topology).
We seek a contradiction. To this end, for every $\beta < \Theta$, set $P_{\beta} = \bigcap_{n=1}^{\infty} P_{\beta,n}$, where $P_{\beta,n}$ are open decreasing sets, and

$$E_n = \bigcup_{\beta < \Theta} (A_{\beta} \times P_{\beta,n}), \quad n \in \mathbb{N}.$$ 

Then $E = \bigcap E_n$ and, in order to show that $E$ is measurable (resp., $BP$-measurable), it suffices to show that so are the sets $E_n$.

Let $G$ be a countable base of open sets in $Y$ and fix $E_n, n \in \mathbb{N}$. For $G \in G$, let

$$B_G = \{ A_{\beta} : G \subset P_{\beta,n} \}.$$ 

Write $B_G = \bigcup B_G$. As $E_n = \bigcup_{G \in G} (B_G \times G)$, the set $E_n$ is measurable (resp., $BP$-measurable), if each $B_G$ is so, which gives the desired contradiction. However, if some $B_G$ is not, then the lemma holds. Indeed, $B_G$ is then a subfamily of $\{ A_{\gamma} : \gamma \in \Gamma \}$ whose union is not $\mu$-measurable (resp., not $BP$-measurable).

**Remark 3.3.** The assumption of regularity of $\mu$ is automatically satisfied if $Y$ is a metric space.

**Theorem 3.4.** Let $Y$ be a 2nd countable topological space, $\mu$ a finite regular Borel measure on $Y$, $X$ a metric space, and $f : Y \to X$ a $\mu$-measurable (resp., $BP$-measurable) map. Then $f$ is essentially separably valued.

**Proof.** If $\mu$ is trivial, there is nothing to prove. Let $\mu(Y) > 0$ and let $\mathcal{B}$ be an open cover of $X$. By the Stone Theorem [13, 4.4.1], there exists an open refinement $\mathcal{D}$ of $\mathcal{B}$ which is $\sigma$-disjoint, i.e., such that $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}^n$, where $\mathcal{D}^n$ are disjoint families in $X$. Given $n \in \mathbb{N}$, there is only a sequence $(\mathcal{D}^n_k, k = 1, 2, \ldots)$ of all sets in $\mathcal{D}^n$ for which $\mu(f^{-1}(\mathcal{D}^n_k)) > 0$. We note that the 2nd countability of $Y$ is not needed here; the fact that any family of disjoint sets of positive measure must be countable is used.

Now, $\mathcal{A}^n = \{ A = f^{-1}(D) : D \in \mathcal{D}^n \setminus \{ \mathcal{D}^n_k : k \in \mathbb{N} \} \}$ is a family of disjoint $\mu$-zero sets in $Y$. Since the union of any subfamily of open sets is open, the union of any subfamily of sets in $\mathcal{A}^n$ is $\mu$-measurable. Hence $\bigcup \{ A : A \in \mathcal{A}^n \}$ is a set of measure zero by Lemma 3.2. Denoting $\mathcal{A} = \bigcup_n A^n$, one gets $\mu(\bigcup \{ A : A \in \mathcal{A} \}) = 0$. As $\mathcal{D}$ is a cover of $Y$, we conclude that $\bigcup \{ f^{-1}(\mathcal{D}^n_k) : n \in \mathbb{N}, k \in \mathbb{N} \}$ is a set of full measure in $Y$. But $\mathcal{D}$ is a refinement, so for every $\mathcal{D}^n_k$ one can find $B \in \mathcal{B}$ containing it. This defines a countable subfamily of $\mathcal{B}$, making an application of Lemma 3.1 possible, which ends the proof.

Let now $f$ be $BP$-measurable. We may assume that the space $Y$ is 2nd category, because otherwise there is nothing to prove. By the Banach Category Theorem [28, 16.1], $Y$ can be represented as a union of a 1st category set $Y^0$.
and its complement $Y_1$, where $Y_0$ is the union of all 1st category open subsets of $Y$. Consider two (disjoint) members of $D^n$, say $D_1$ and $D_2$, whose inverse images by $f$ are both 2nd category. As $f$ is $BP$-measurable, we can write $f^{-1}(D_1) = (O_1 \setminus I_1) \cup J_1$ and $f^{-1}(D_2) = (O_2 \setminus I_2) \cup J_2$, where $O_1, O_2$ are open sets and the other four sets are 1st category. Observe that $O_1 \cap O_2$ are disjoint nonempty open sets in $Y_1$. Consequently, given $n \in \mathbb{N}$, the family of all sets in $D^n$ whose inverse images by $f$ are 2nd category is countable and, therefore, can again be written $(D^n_k)_{k=1,2,\ldots}$. This time, to conclude that there was only a countable number of such sets, we used the fact that $Y_1$ is 2nd countable.

Hence $A^n = \{A = f^{-1}(D) : D \in D^n \setminus \{D^n_k : k \in \mathbb{N}\}\}$ is a family of disjoint 1st category sets. By Lemma 3.2 again, the union of the sets in $A^n$ is 1st category. Keeping the notation from the first part of the proof, the union of the sets in $A$ is 1st category and so $\bigcup\{f^{-1}(D^n_k) : n \in \mathbb{N}, k \in \mathbb{N}\}$ is a residual set in $Y_1$. This permits it to finish the proof as in the first part.

**Remark 3.5.** Assuming appropriate separation axioms on the topology of $Y$, there exist results stronger than Theorem 3.4, see e.g. [15] and [16]. The main reason for its inclusion in the form above is the desire to expose the duality of our proof of the ‘Baire counterpart’ with the proof of the ‘measure part’. The latter proof can be treated as more or less standard. It can be traced back at least as far as to the proof of Th. 2 in [3, Appendix 3].

## 4 Exhaustivity

Recall that $X$ is a Hausdorff topological Abelian group. Let $m : \mathcal{P} \to X$ be (a finitely additive measure or) an additive map, that is, a map such that $m(A \cup B) = m(A) + m(B)$ for disjoint sets $A$ and $B$. It is called exhaustive whenever $m(E_n) \to 0$ for each sequence of disjoint sets $(E_n) \in \mathcal{P}$.

We first prove a result showing that both, $BP$-measurable and $\chi$-measurable additive maps enjoy a sort of ‘weak exhaustivity’ property.

**Proposition 4.1.** Let $m : \mathcal{P} \to X$ be a Haar measurable (resp., $BP$-measurable) additive map. Then $m(\{n\}) \to 0$.

**Proof.** By embedding $X$ into a product of normed groups, we may assume that $X$ is metric. Let $U$ be a neighborhood of 0 in $X$ and find another neighborhood of 0 in $X$ such that $V - V \subset U$. As $m$ is essentially separable by Theorem 3.4, there is a sequence $(x_n)$ of points in $X$ such that the inverse
images of the translates, \( m^{-1}(x_n + V) \), cover \( \mathcal{P} \) apart, perhaps, from a measure zero (resp. 1st category) subset of \( \mathcal{P} \). Hence one of them, say \( m^{-1}(x_k + V) \), is of positive \( \chi \)-measure (resp., 2nd category). By the Steinhaus (resp., Piccard) Theorem, the family \( \mathcal{A} = m^{-1}(x_k + V) \Delta m^{-1}(x_k + V) \) is a neighborhood of \( \emptyset \) in \( \mathcal{P} \).

\( \mathcal{A} \) being a neighborhood of \( \emptyset \), there exists \( m \in \mathbb{N} \) so large that \( \{n\} \in A \) for \( n > m \). One therefore has \( \{n\} = E \Delta F \) with \( m(E) \in x_k + V \) and \( m(F) \in x_k + V \). Since \( \{n\} \) is a singleton, \( E \subset F \) or vice versa. Suppose the latter; then \( \{n\} = E \Delta F = m(E) \) is of positive \( \chi \)-measure (resp., 2nd category). By the Steinhaus (resp., Piccard) Theorem, the family \( \mathcal{A} = m^{-1}(x_k + V) \Delta m^{-1}(x_k + V) \) is a neighborhood of \( \emptyset \) in \( \mathcal{P} \).

Proposition 4.2. If \( m : \mathcal{P} \to X \) is \( BP_r \)-measurable, then it is exhaustive.

Proof. Let \( (E_n) \) be a sequence of disjoint sets in \( \mathcal{P} \). Define the map \( j : \mathcal{P}(\mathbb{N}) \to \mathcal{P} \) by putting for each \( F \subset \mathbb{N} \)

\[ j(F) = \bigcup_{n \in F} E_n. \]

To avoid confusion, denote the domain Cantor group of \( j \) by \( \mathcal{R} \). The map \( j \) is a continuous injection from \( \mathcal{R} \) into \( \mathcal{P} \) and, in fact, a homeomorphism onto its image, \( Z \) say, because \( \mathcal{R} \) is compact. By the assumption on \( m \), if \( O \subset X \) is open, then \( m^{-1}(O) \cap Z \) is \( BP \)-measurable relative to \( Z \). Hence its inverse image by \( j \) is still \( BP \)-measurable in \( \mathcal{R} \). Thus, \( mj : \mathcal{R} \to X \) is \( BP \)-measurable. By ‘Baire part’ of Proposition 4.1, \( mj(\{n\}) = m(E_n) \to 0 \). \( \Box \)

Recall that a (finite positive) Borel measure \( \mu \) on a Hausdorff space \( Y \) is said to be Radon measure if it is inner regular with respect to compact sets [16]. A set \( E \) in \( Y \) is universally measurable, if it is measurable for every Radon measure \( \mu \) on \( Y \). A function \( f \) from \( Y \) into a topological space \( X \) is universally measurable, if for every open subset \( O \) in \( X \), the set \( f^{-1}(O) \) is universally measurable in \( Y \).

For historical reasons, let us mention that the notion of universal measurability was defined precisely as a measure analogue of \( BP_r \)-measurability (by Szpilrajn-Marczewski in [34] under the name of absolute measurability).
Proposition 4.3. If \( m : \mathcal{P} \to X \) is universally measurable, then it is exhaustive.

Proof. Keeping the notation from the proof of the preceding Proposition 4.2, let \( \chi \) be the Haar measure on \( \mathcal{R} \). Further, let \( \mu = \chi j^{-1} \) be its image measure on \( \mathcal{P} \) defined the usual way. Since \( \mu \) is a Radon measure ([16, 418 I]) and \( m \) is universally measurable, we conclude that \( m \) is \( \chi j^{-1} \)-measurable. This means that \( mj : \mathcal{R} \to X \) is \( \chi \)-measurable. It follows from the Proposition 4.1 that \( mj(\{n\}) = m(E_n) \to 0. \)

5 Countable additivity

Again, \( X \) is a Hausdorff topological Abelian group throughout. Let \( \sum_n x_n \) be a subseries convergent series in \( X \). Then the canonical measure \( m : \mathcal{P} \to X \) (or canonical map \( m : K \to X \)) connected with it, is defined by

\[
m(E) = \sum_{n \in E} x_n \quad \text{for} \quad E \subset \mathbb{N}.
\]

An additive map \( m : \mathcal{P} \to X \) is a measure (i.e., countably additive) iff it is continuous iff it is the canonical map of the series \( \sum_n x_n \), where \( x_n = m(\{n\}) \).

A situation of interest arises in connection with theorems of Orlicz-Pettis type. On \( X \) two Hausdorff group topologies \( \alpha \) and \( \beta \) are considered, with \( \alpha \subset \beta \). One knows that a series \( \sum_n x_n \) is subseries convergent in \( (X, \alpha) \). One seeks criteria on \( \beta \) so that the series is also convergent in \( (X, \beta) \). The canonical map of the series, \( m : \mathcal{P} \to X \), is \( \alpha \)-continuous, but is, a priori, only an additive map into \( (X, \beta) \).

The following theorem goes back to [25], [29] and [17].

Theorem 5.1. Let \( \alpha, \beta \) be Hausdorff group topologies on \( X \), with \( \alpha \subset \beta \) and \( \beta \) sequentially complete. Suppose the identity map \( \iota : (X, \alpha) \to (X, \beta) \) is universally measurable. If \( \sum x_n \) is subseries convergent in \( (X, \alpha) \), then it is so in \( (X, \beta) \).

Proof. It is sufficient to show that the canonical continuous map \( m : \mathcal{P} \to X \) associated with the series in \( (X, \alpha) \) is exhaustive into \( (X, \beta) \). Indeed, then \( \sum_n x_n \) satisfies the Cauchy condition for summability in \( (X, \beta) \) and therefore is subseries convergent there to the same limit as in \( (X, \alpha) \). Consider a Radon measure \( \mu \) on \( \mathcal{P} \) and its image measure \( \nu = \mu m^{-1} \) in \( (X, \alpha) \). As \( \nu \) is Radon ([16, 418 I]), the identity \( \iota \) is \( \nu \)-measurable. Consequently, \( \nu \) is \( \mu \)-measurable or, which is the same, \( m \) into \( (X, \beta) \) is \( \mu \)-measurable. We have shown that \( m : \mathcal{P} \to (X, \beta) \) is universally measurable and, therefore, exhaustive by Proposition 4.3. \( \square \)
Problem 5.2. Is the ‘dual statement’ to the above theorem, in which one assumes the identity \( \iota \) to be \( BP_r \)-measurable, true?

Remark 5.3. For the origins of the Orlicz-Pettis Theorem, see [20] and [14]. As for the Problem 5.2, there exist some results in this direction. For instance, in [10, Theorem 2.1] the measurability assumption about \( \iota \) is even weaker, but a very strong completeness of \( \beta \) is needed.

If \( \iota \) is Borel measurable, there is no problem. Though this case is already covered since Borel measurable maps are universally measurable, it leads us to a theorem of N.J.M. Andersen and J.P.R. Christensen [1]. But before we state the theorem, let us digress and present another double result whose ‘Baire part’ will be needed.

Consider an equivalence relation \( \sim \) in a product \( Y = \prod_{i \in I} Y_i \) defined by:

\[
a \sim b \quad \text{if the set } \{i \in I : a(i) \neq b(i)\} \text{ is finite.}
\]

A subset \( A \) of \( Y \) is a tail set if \( y \in A \) and \( y \sim z \) implies \( z \in A \). A function \( f : Y \to X \) is compatible with \( \sim \) if it is constant on equivalence classes \( y \sim \) for \( y \in Y \).

Suppose now that \( Y \) is a product of probability spaces with the product probability measure \( \mu \) (resp., \( Y \) is a product of Baire spaces each of which has a countable pseudo-base). For the record, a family \( B \) of non-empty open sets in a topological space is a pseudo-base, if every non-empty open set contains at least one member of \( B \). The so-called 0-1 law of probability theory (resp., topological 0-1 law) says that, if \( A \subset Y \) is a \( \mu \)-measurable (resp., \( BP \)-measurable) tail set, then \( \mu(A) = 0 \) or 1 (resp., \( A \) is 1st category or residual subset of \( Y \)), see [19, Section 46(3)], [27, Theorem 4].

Proposition 5.4. Let \( Y \) be a product of Baire spaces, each of which has a countable pseudo-base (resp., product of probability spaces with product measure \( \mu \)), and let \( X \) be a metric space. Assume \( f : Y \to X \) is \( BP \)-measurable (resp, \( \mu \)-measurable) and compatible with \( \sim \). Then \( f \) is constant on a residual (resp, \( \mu \)-measure one) subset of \( Y \).

Proof. Let \( B_k \) be an open cover of \( X \) by balls of radius \( 1/k \). By the already invoked Stone Theorem [13, 4.4.1], there exists an open refinement \( D \) of \( B_k \) with \( D = \bigcup_{n=1}^{\infty} D^n \), where \( D^n \) are disjoint families in \( X \). Given \( n \in \mathbb{N} \), let \( E^n \) be the subfamily of all sets \( D \) in \( D^n \) for which \( f^{-1}(D) \) is 2nd category (resp., of positive measure) and therefore residual (resp., of measure one). \( A^n = \{A = f^{-1}(D) : D \in D^n \setminus E^n \} \) is a disjoint family of 1st category (resp., measure zero) sets in \( Y \) and therefore, by Lemma 3.2, the union \( Y_{0}^{n} = \bigcup \{A : A \in A^n\} \) is 1st category (resp., measure zero) set. Hence \( Y_{0} = \bigcup \{Y_{0}^{n} : n \in \mathbb{N}\} \) is 1st
category (resp., measure zero) in $Y$. As $D$ is a cover, $E = \bigcup_{n=1}^{\infty} E^n$ is a family covering $Y_1 = Y \setminus Y_0$, a residual (measure one) set in $Y$. By choosing for each $D \in E$ a ball in $B_k$ containing it, we conclude that:

For each open cover $B_k$ of $X$ by balls of radius $1/k$ there exists a 1st category (resp., measure one) set $Y_0(k)$ and a subfamily $B_k^*$ of $B_k$ covering

$Y_1(k) = Y \setminus Y_0(k)$ such that for each $B \in B_k^*$, $f^{-1}(B)$ is residual (resp., measure one) in $Y$.

In particular, as $Y$ is a Baire space by [27, Theorem 3], we see that the families $B_k^*, k \in \mathbb{N},$ are nonempty. Choose a ball $B_{2n_1}$ in $B'_2$ and note that $f^{-1}(B_{2n_1})$ is a residual (resp., measure one) subset of $Y$. On the second step, choose a ball $B_{3n_2} \in B'_3$. Again, $f^{-1}(B_{3n_2})$ is residual (resp., measure one) in $Y$ and so is $f^{-1}(B_{2n_1}) \cap f^{-1}(B_{3n_2})$. Continuing, we will find a nested sequence of balls $(B_{kn_k})$ with radius $1/k \to 0$ and such that $Z = \bigcap \{f^{-1}(B_{kn_k}) : k \in \mathbb{N}\}$ is a residual (resp., measure one) set in $Y$. Pick $y \in Z$. Then $x = f(y) = \bigcap_{k=1}^{\infty} B_{kn_k}$ and $f^{-1}(x) = \bigcap_{k=1}^{\infty} f^{-1}(B_{kn_k}) = Z$. Hence $f$ is constant on a residual (resp., measure one) set $Z$. 

Here is the Andersen-Christensen Theorem in its generalized (and corrected, see Remark 5.6 below) form.

**Theorem 5.5.** Let $X$ be sequentially complete and $m : \mathcal{P} \to X$ a $BP_1$-measurable additive map. Then $m$ is the canonical measure of the subseries convergent series $\sum_n m\{n\}$.

In the arguments that follow, the structure of the Cantor group $K$ as product $\{0,1\}^{\mathbb{N}}$ and the continuity of $m$ on $K$ rather than the countable additivity of $m$ on $\mathcal{P}$ will be exploited. For this reason, it is more convenient to think in terms of $m : K \to X$ and change the notation accordingly. The group operation of addition $mod.2$ in $K$ is denoted by $+$ and $a, b, c, \ldots$ are the elements of $K$. Then $F$ corresponds to $\mathcal{F}$, $0$ to $\emptyset$, $n$ to the singleton $\{n\}, ab$ to $A \cap B$, $a - ab$ to $A \setminus B$, and $e$ will be reserved for $\{1, 1, \ldots\}$.

**Proof.** Given $m$, by the Proposition 4.2 and the sequential completeness of $X$, the series $\sum_n m\{n\}$ is subseries convergent. Its canonical map $m'$ from $K$ to $X$ is continuous. Consider $m'' = m - m'$ on $K$. It is obvious that $m''$ is an additive map vanishing on $F$. We claim that it is $BP_1$-measurable. As usual, we may embed $X$ into a product of normed groups, and therefore it will be enough to assume that in the next step of the proof $X$ is metric.

Let $Z \subset K$. We need to show that $(m - m')|Z$ is $BP_1$-measurable. Clearly, $m|Z$ is $BP_1$-measurable and $m'|Z$ is continuous. Using Theorem 3.4, find $H_1 \subset Z$, 1st category in $Z$, such that $m|(Z \setminus H_1)$ is separably valued. As $m|(Z \setminus H_1)$ is $BP_1$-measurable, by [24, Section 32, II] there exists a set $H_2$,
first category in $Z \setminus H_1$ and therefore also in $Z$, such that $m(Z \setminus (H_1 \cup H_2))$ is continuous. Hence $m - m'$ is continuous apart of 1st category set which can taken to be an $F_\sigma$ subset of $Z$. Let $O$ be an open set in $X$. Observe that $(f - g)^{-1}(O)$ can be written as a disjoint union of a 1st category set in $Z$ and an open subset of a $G_\delta$ in $Z$ and so a $G_\delta$ subset of $Z$. It follows by [24, Section 11.IV.2] that the inverse image under consideration is a $BP$-measurable subset of $Z$, as needed.

The proof will be complete, if we can show that $m''$ is identically zero. This will be done in the next Proposition (the proposition is implicit in the original proof of the theorem in [1]).

**Remark 5.6.** Andersen and Christensen claim that the assumption of sequential completeness of $X$ is not needed. However, without any completeness assumption on $X$, the countably additive and therefore continuous function $m'$ takes values in the completion $\hat{X}$. Consequently, $m - m'$ also takes values in $\hat{X}$ and we do not know whether it is $BP_r$-measurable, a fact needed to conclude that it vanishes. Assuming Borel measurability does not improve the situation.

**Proposition 5.7.** Let $X$ be normed, and $m : K \to X$ be an additive $BP_r$-measurable map. If $m|F = 0$, then $m$ is identically zero.

**Proof.** By Proposition 5.4, $m$ is constant on a dense $G_\delta$ subset $A$ of $K$. Let $x$ be that constant. Observing that $A \cap e + A$, as an intersection of dense $G_\delta$ sets, is nonempty, we deduce that $e = a + b$ with $a \in A$, $b \in A$ and $ab = 0$ (i.e. $a$ and $b$ are disjoint as elements of $P$). It follows that $m(e) = 2x$. At this point, we need the following lemma (Lemma after Theorem 2 in [7, p. 247]):

If $A$ is a dense $G_\delta$ subset of $K$, then there exist $a, b, c$ in $A$ such that $c = a + b$ with $a, b$ disjoint.

So, if the lemma holds, $x = 0$ and $m(\mathbb{N}) = 0$. Let now $a$ be an arbitrary element of $K$. If $a \in F$, then $m(a) = 0$. If $a \notin F$, then denote by $A$ the support of $a$ in $\mathbb{N}$, i.e., $\{n \in \mathbb{N} : a(n) \neq 0\}$. The restriction of $m$ to $K_a = \{0, 1\}^A$ is again $BP_r$-measurable. Hence, by the proof above, $m(a) = 0$.

It remains to prove the lemma. Here is Christensen’s proof of it. Consider the maps $g, h : K \times K \to K$ defined by

$$g(a, b) = a - ab \text{ and } h(a, b) = ab.$$  

As $g, h$ are surjective, open and continuous, $g^{-1}(A)$ and $h^{-1}(A)$ are dense $G_\delta$ sets in $K \times K$. Choose

$$(a, b) \in g^{-1}(A) \cap h^{-1}(A) \cap (A \times K)$$

and put $x = a$, $y = g(a, b)$ and $z = h(a, b)$. 

□
Problem 5.8. Is the ‘universal measurability’ dual of Theorem 5.5 true?

Remark 5.9. J. P. R. Christensen (assuming the Continuum Hypothesis) claims having an example that solves Problem 5.8 in the negative, see [8, Theorem 6.1] and the example following it. However, if true, his claim would need a better proof.

6 Analogy breaks

We will need the following consequence of the main result of Section 3.

Corollary 6.1. Let $Y$ be second countable Baire space, $X$ a metric space, and $f$ a function from $Y$ to $X$. The following conditions are equivalent.

(a) $f$ is BP-measurable.

(b) $f$ is essentially separable.

(c) $f$ is continuous apart from a 1st category set.

Proof. (a) implies (b) is Theorem 3.4. (a) implies (c). There exists 1st category set $E \subset Y$ such that the restriction $f|(Y \setminus E)$ has separable image. As $Y$ is Baire, $Y \setminus E$ is a dense. Let $O$ be an open set in $X$. By [24, §11, V, Theorem 2], if $f^{-1}(O)$ is BP-measurable in $Y$, then $f^{-1}(O) \cap (Y \setminus E)$ is so relative to $Y \setminus E$. This means that the restriction $f|(Y \setminus E)$ is BP-measurable relative to $Y \setminus E$. By the proof of the necessity part of [24, §32, II], we can find a 1st category set $F$ in $Y \setminus E$ such that $f|(Y \setminus (E \cup F))$ is continuous. But $F$ is then also 1st category in $Y$, i.e., (c) holds. Now, the sufficiency part of the just invoked proof of Kuratowski gives (c) implies (a).

The Corollary above overlaps with the main theorem of [12] (our $Y$ does not have to be Čech complete).

Theorem 6.2. Suppose an additive map $m : K \to X$ is BP-measurable. Then $m$ is exhaustive.

Proof. By embedding $X$ into a product of normed groups, we may assume that $X$ is metric. Suppose that $m : K \to X$ is not exhaustive. Then there exists a disjoint sequence of $a_n \in K$ such that $m(a_n) \neq 0$. In view of Corollary 6.1, there exists a dense $G_\delta$ subfamily $A$ of $K$ on which $m$ is continuous. As the set $F \cup \{a_n : n \in \mathbb{N}\}$ is countable, we may assume that $A$ is translation invariant with respect to its elements. Let $b \in A$ and define $b'_n = b + a_n + ba_n$ and $b''_n = b + ba_n = b \sim ba_n$. Both $b'_n$ and $b''_n$ are in $A$, $b'_n \to b$ and $b''_n \to b$.
there. Hence $m(b'_n) - m(b''_n) \to 0$. On the other hand, $b'_n$ and $a_n$ are disjoint and $b'_n = b''_n + a_n$. Consequently, $m(b'_n) - m(b''_n) = m(a_n) \not\to 0$. A contradiction.

The result sharpens Proposition 4.2. But from our point of view, it is in a sense ‘too good’—it is at this point that the analogy between measure and category breaks down.

**Example 6.3.** Denote by $A$ the set of points $a = (\varepsilon_n) \in K$ for which $\lim_n \frac{1}{n} \sum \varepsilon_n = \frac{1}{2}$, i.e., the set of normal numbers. By Borel’s normal number theorem [4, Theorem 1.2], $\chi(A) = 1$. Now switch the interpretation from $K$ to $\mathcal{P}$ (and so $A$ will correspond to $A$). Identifying sets with their characteristic functions, consider $\text{lin}(A)$ and $\text{lin}(\mathcal{P})$. Let $\mathcal{B}$ be a set of linearly independent vectors contained in $\mathcal{A}$ which is maximal with respect to inclusion in $\mathcal{A}$. As easily checked, $\mathcal{B}$ is a Hamel basis of $\text{lin}(A)$. Let $\mathcal{H}$ be a Hamel basis consisting of vectors in $\mathcal{P}$, containing $\mathcal{B}$, and spanning $\text{lin}(\mathcal{P})$. Define a linear functional $m$ on $\text{lin}(\mathcal{P})$ such that $m$ is zero on $\text{lin}(A)$, and is unbounded on $\mathcal{H}$. This is possible because the co-dimension of the linear subspace $\text{lin}(A)$ in $\text{lin}(\mathcal{P})$ is infinite and the coefficients of basic vectors in $\mathcal{H} \setminus \mathcal{B}$ can be chosen arbitrarily. Consider the restriction of $m$ to $\mathcal{P}$, keeping the notation $m$. The additive map $m$ is measurable with respect to $\chi$ (since it is $\chi$-almost everywhere continuous), but it is unbounded, and so it is not exhaustive on $\mathcal{P}$.

**Remark 6.4.** The technique used to prove Theorem 6.2 is a modification of an argument due to Andersen and Christensen (see the beginning of the proof of [1, Theorem 1]). Example 6.3 is a somewhat polished result of Constantinescu [9, Proposition 16].

**References**


