The University of Mississippi
Department of Mathematics

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INTRODUCTION

This present volume is a collection of articles presented during the conference to honor the memory of Wladyslaw Orlicz (1903-1990). This conference was organized by the Department of Mathematics of the University of Mississippi.

Wladyslaw Orlicz, one of founders of modern functional analysis and the last surviving member of the Banach Mathematical School, died on August 9, 1990. With him a chapter of functional analysis passed into history.

Several of his students, some of whom are now at the University of Mississippi, others at various schools in the country, decided to meet to review these chapters of mathematics they studied under the influence of Professor Orlicz. With this in mind a number of colleagues were called upon to meet together. In spite of a short notice, we are pleased that many distinguished mathematicians accepted our invitation and were able to attend.

We were lucky enough to have the support of the University of Mississippi. We owe much thanks for this support to Dr. Michael Dingerson, Dean of the Graduate School and Dr. Dale Abadie, Dean of the College of Liberal Arts. In addition, it is worthwhile to note that owing to the strong Polish participation in the meeting, funds were generously provided by the Kosciuszko Foundation of America. We express our deep appreciation.

We dedicate the present volume to the memory of Professor Wladyslaw Orlicz.

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Commutativity of certain algebras using a Mazur-Orlicz idea.

by

Gerard Buskes

Let $E$ be a Riesz space (= vector lattice). For $x, a \in E$ we define

$$\|x\|_a = \inf \{ \lambda \in \mathbb{R}^+ \mid |x| \leq \lambda |a| \},$$

where it is understood that $\|x\|_a = \infty$ if the set on the right hand side is empty. It is easily seen that $\|\cdot\|_a$ defines a subadditive, positively homogeneous and supremum preserving map $E \to \mathbb{R}^+ \cup \{ \infty \}$.

In the usual way $\|\cdot\|_a$ defines a topology on $E$, the $\{a\}$-topology. For $A \subset E$, a subset $S$ of $E$ is called $A$-open if it is a member of the $\{a\}$-topology for all $a \in A$. The $A$-open sets form a topology, the $A$-topology. $E$ is said to be small if there exists a countable $\mathbb{Q}$-linear sublattice (that is, a that is, a countable vector space over the rationals and sublattice ) $A$ of $E$ such that the $A$-closure of $A$ equals $E$. For instance, $c$ is small, but $c_0$ is not small.

On any Riesz space $E$ there exist subadditive, positively homogeneous and supremum preserving maps $E \to \mathbb{R}^+ \cup \{ \infty \}$ ($\|\cdot\|_a$ for every $a \in E$ is an example).

Small Riesz spaces are special in that one can actually construct such maps taking finite values only (see Lemma 2.1 in [2]). From now on let $E$ be a small Riesz space and let $e \in E$. Take a subadditive, positively homogeneous and supremum preserving map $q: E \to \mathbb{R}$ with $q(e) = 1$. The following trick originated in the work by Mazur and Orlicz ([3]). Let $a_1 \in E$ and define

$$p_1(x) = \lim_{\lambda \to 1^-} q(x + \lambda a_1) - \lambda q(a_1) \quad (x \in E).$$

Then $p_1$ is a finite valued, sublinear, supremum preserving map $E \to \mathbb{R}$ and

$$-q(-x) \leq -p_1(-x) \leq p_1(x) \leq q(x) \text{ for all } x \in E.$$

Also, $p_1(a_1) = q(a_1)$ and $p_1(-a_1) = -q(a_1)$.

Since $E$ is small there exists a countable $\mathbb{Q}$-linear sublattice
A= \{e=a_1,a_2,\ldots\ldots\ldots\} \text{ of } E \text{ such that the } A\text{-closure of } A \text{ in } E \text{ equals } E.

Define inductively

\[ p_{n+1}(x) = \lim_{\lambda \to 0} p_n(x+\lambda a_{n+1}) - \lambda p_n(a_{n+1})(x \in E), \]

and \[ \varphi(x) = \lim_{n \to \infty} p_n(x) \quad (x \in E). \]

Then \( \varphi(e) = q(e) = 1 \) and by showing that

\[ \{x \in E \mid \varphi(x) + \varphi(-x) = 0\} = E, \]

one proves that \( \varphi \) actually is a Riesz homomorphism. The above argument shows (see 2.2 in [2]):

Theorem 1. The Riesz homomorphisms \( E \to \mathbb{R} \) separate the points of \( E \).

A nice illustration of the technique of small Riesz spaces is provided by looking at the automatic commutativity of almost f-algebras. That almost f-algebras are commutative was recently proved by Bernau and Huijsmans (in [1]). They term their proof alternative, elementary, intrinsical and representation-free. Their proof is several pages long and somewhat technical. The following ten line proof of a slightly more general statement was communicated to me by A. van Rooij.

Let \( E \) be an Archimedean Riesz space and let \( * \) be a multiplication on \( E \) which renders \( E \) an algebra. For \( A,B \subseteq E \) define \( A*B=\{a*b \mid a \in A, b \in B\} \). Suppose \( E \) has the properties

1. \( A*B \) is order bounded when \( A,B \subseteq E \) are.
2. If \( f*g = 0 \) then \( f*g = 0 \)

Theorem 2. \( E \) is commutative. In particular, every almost f-algebra is commutative.

Proof. Take \( a,b \in E \). To show that \( a*b = b*a \), we may assume that \( E \) is a small
Riesz space. Ignoring the algebra structure, let D by any small Riesz
subspace of E containing a,b as well as an order unit. We are done if
\( \psi(a \ast b) = \psi(b \ast a) \) for all Riesz homomorphisms \( \psi: E \to \mathbb{R} \). To this end take a
Riesz homomorphism \( \psi \) and define \( \omega(x,y) = \psi(x \ast y) \) for all \( x,y \in D \) and apply the
following.

**Lemma.** Let \( X \) be a compact Hausdorff space, \( D \) a dense Riesz subspace of \( C(X) \)
with \( l \in D \). Let \( \omega: D \times D \to \mathbb{R} \) be bilinear and such that

1. \( \{ \omega(f,g) \mid f \in A, g \in B \} \) is order bounded if \( A,B,C,D \) are, 
2. \( f \land g = 0 \Rightarrow \omega(f,g) = 0. \)

Then \( \omega(f,g) = \omega(g,f). \)

**Proof.** \( \omega \) induces a map \( \omega^\circ \in C(X \times X)^{\sim} \). Let \( L \subset C(X \times X) \) be the closed linear
hull of \( \{ f \otimes g \mid f,g \in D, f \land g = 0 \} \). It follows that \( L = \{ f \in C(X \times X) \mid f|_z = 0 \} \)
where \( z \) is the diagonal of \( X \times X \). Then \( \omega^\circ|_L = 0 \). In particular we find

\[ \omega^\circ(f \otimes g - fg \otimes 1) = 0 \]

and thus \( \omega(f,g) = \omega^\circ(gf \otimes 1) = \omega^\circ(gf \otimes 1) = \omega(g,f). \)


The atomic space problem and related questions for F-spaces

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In this note we consider some open questions concerning linear metric spaces. Although we will prove one or two minor results, our main objective is to discuss the relationship between certain problems related to the fundamental atomic space problem. We will freely use standard notions in the theory of complete metric linear spaces (F-spaces); see [7] and [14].

1. The atomic space problem.

Probably the most fundamental problem concerning F-spaces is:

PROBLEM 1: THE ATOMIC SPACE PROBLEM. Does every infinite-dimensional F-space contain a proper closed infinite-dimensional subspace?

This question can be traced to Pelczynski in the early sixties, but perhaps it has been around longer [9] (see also [7] and [14]). It is such an obvious question that it seems quite amazing that the answer is still unknown. We will say that an infinite-dimensional F-space is atomic if it contains no proper closed infinite-dimensional subspace. Of course an atomic space must be separable.

In order to discuss this problem further let us define an F-space $X$ to be minimal if there is no strictly weaker Hausdorff vector topology on $X$. In [4] it is shown that the space $\omega$ of all sequences is minimal. In fact any minimal space with a basis is isomorphic to $\omega$, by simply noting that the F-space topology must coincide with the topology induced by the biorthogonal functionals. Also note that a closed subspace of a minimal space is minimal. Thus a minimal space which contains a basic sequence contains a subspace isomorphic to $\omega$; in particular a minimal quasi-Banach space contains no basic sequence. The following result is shown in [4] and [8]:

THEOREM 1.1 [8]. Let $X$ be a non-minimal F-space. Then $X$ contains a basic sequence $(x_n)$. Further $(x_n)$ can be chosen to be regular i.e. bounded away from zero.

Plainly an atomic space must be minimal. In fact it must also be quotient-minimal i.e. every quotient must also be minimal [3].

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THEOREM 1.2. Let $X$ be an F-space. Then $X$ contains a basic sequence if and only if it contains a decreasing sequence $(L_n)$ of infinite-dimensional closed subspaces so that $\bigcap L_n = \{0\}$.

PROOF: One direction is clear. If $(x_n)$ is a basic sequence simply let $L_n = [x_k : k \geq n]$. Conversely if $(L_n)$ is such a sequence of subspaces we can assume that $L_n \neq L_{n+1}$ for all $n$. Consider the topology $\tau$ generated by the F-seminorms $x \to d(x, L_n)$ for $n \in \mathbb{N}$. If $\tau$ coincides with the original topology then picking $x_n \in L_n \setminus L_{n+1}$ with $x_n \neq 0$ it is simple to verify that $(x_n)$ is a basic sequence and $[x_n] \sim \omega$. If $\tau$ is a strictly weaker topology then $X$ is non-minimal and so contains a basic sequence.

PROBLEM 2: THE BASIC SEQUENCE PROBLEM. Does every infinite-dimensional F-space contain a basic sequence? Does every infinite-dimensional quasi-Banach space contain a basic sequence?

This in turn relates closely to (cf. [2]):

PROBLEM 3: THE MINIMAL SPACE PROBLEM. Let $X$ be an infinite-dimensional minimal F-space (i.e. such that there is no strictly weaker Hausdorff vector topology on $X$). Must $X$ be isomorphic to $\omega$, the space of all sequences? Is there an infinite-dimensional minimal quasi-Banach space?

Obviously Problems 2 and 3 coincide for quasi-Banach spaces. In this context we note that the results of Bastero [1] on stable p-Banach spaces show that no subspace of $L_p[0,1]$ $(0 < p < 1)$ can be minimal (or atomic). It is apparently unknown, however, whether $L_0[0,1]$ has a minimal subspace.

THEOREM 1.3. Let $X$ be an arbitrary infinite-dimensional F-space. Then either $X$ contains an atomic subspace or $X$ contains a subspace $F$ of finite dimension so that $X/F$ contains a basic sequence.

PROOF: We may suppose that $X$ is separable. Let $\mathcal{L}$ be a maximal collection of infinite-dimensional closed subspaces of $X$ which is closed under finite intersections. Let $F = \bigcap \{L : L \in \mathcal{L}\}$. If $\dim F = \infty$ then $F$ is atomic. If not then $\dim F < \infty$. Since $X$ is separable, by Lindelof's theorem we can pick a sequence $V_n \in \mathcal{L}$ with $F = \bigcap V_n$. Let $L_n = \bigcap_{i=1}^n V_i$. Then $(L_n)$ decreases and $\bigcap L_n = F$. Clearly by Theorem 1.2 $X/F$ contains a basic sequence.

COROLLARY 1.4. Every infinite-dimensional quotient-minimal quasi-Banach space contains an atomic subspace.
2. Partial results and comments.

We would like first to remark that the problems above seem to have a fundamentally different character if one restricts attention to quasi-Banach spaces. Let us first discuss the problems in the full generality of F-spaces. We start with the recent intriguing example due to Reese [10]:

**Theorem 2.1 (Reese).** There exists an F-space $X$ with a sequence of finite-dimensional subspaces $V_n$ so that $\dim V_n \geq n$ and so that $x_n \in V_n$ for every $n$ and $x_n \neq 0$ for infinitely many $n$ then $[x_n] = X$.

Following [10] let us call such a space $X$ almost atomic. It is clear that an almost atomic space has trivial dual (and, with a little bit more work any compact operator on $X$ is zero). Further a quotient of an almost atomic space remains almost atomic.

It is rather unclear to the author just how unusual almost atomic spaces are, particularly in the general setting of F-spaces. It is natural thus to ask the following:

**Problem 4.** Is $L_0[0,1]$ almost atomic?

It is important to stress that Reese’s example is not a quasi-Banach space and it does not appear to be an easy task to make an almost atomic quasi-Banach space.

**Problem 5.** Does there exist an almost atomic quasi-Banach space?

Recently, together with one of my students, S.C. Tam, I have been considering this problem in some special cases. We can show for example that $L_p[0,1]$ ($0 < p < 1$) has no almost atomic subspace.

Now let’s consider another variation. Suppose we try to find a minimal quasi-Banach space without an atomic subspace (e.g. if we believe that no atomic spaces exist). A glance at Theorem 1.3 shows us that we can look for a space $X$ with a finite-dimensional subspace $F$ so that $X/F$ has a basis.

**Theorem 2.2.** Let $X$ be a quasi-Banach space and suppose $F$ is a finite-dimensional subspace so that $X/F$ has a basis. Then $X$ is minimal if and only if the quotient map $Q : X \to X/F$ is strictly singular.

**Proof:** First suppose $Q$ is not strictly singular. Then obviously $X$ contains a basic sequence. Conversely suppose $Q$ is strictly singular. Suppose $(x_n)$ is a basic sequence in $X$. Then there is a block basic sequence $(y_n)$ of $(x_n)$ with $\|y_n\| = 1$ and such that $\|Qy_n\| < 2^{-n}$. Thus there exist $f_n \in F$ so that $\|y_n - f_n\| < 2^{-n}$. By standard perturbation arguments $(f_n)_{n \geq N}$ is basic for some $N \geq 1$; but this contradicts the fact that $F$ is finite-dimensional.  

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Now it is simple to show that if we can find a minimal space of this type then we can suppose $\dim F = 1$. Finally we can even wonder whether this can be done with $X/F$ a Banach space:

**Problem 6.** Does there exist a minimal quasi-Banach space $X$ with a one-dimensional subspace $F$ so that the quotient $X/F$ is a Banach space (e.g. $\ell_1$) and the quotient map is strictly singular?

There are many examples of non-locally convex quasi-Banach spaces $X$ so that for a one-dimensional subspace $F$ we have $X/F$ isomorphic to a Banach space (see e.g. [5],[12], [13]). However, the additional requirement here that the quotient map is strictly singular does not appear to be satisfied in any known example.

3. The Hahn-Banach extension problem.

The author’s interest in minimal spaces and basic sequences originated in the study of the Hahn-Banach Extension Property for $F$-spaces [4]. We say that a topological vector space $X$ has the Hahn-Banach Extension Property or (HBEP) if whenever $V$ is a linear subspace of $X$ and $f$ is a continuous linear functional on $V$ then $f$ has a continuous extension $F$ on $X$. The main result of [4] is that an $F$-space with (HBEP) is locally convex. However, (cf. [7]) a topological vector space with (HBEP) need not be locally convex. At the meeting, Przemo Kranz called my attention again to:

**Problem 7.** Let $X$ be a metrizable topological vector space with (HBEP); is $X$ locally convex?

It should be mentioned that Ribe [11] proved some partial results in this direction. He showed that if $X$ is metrizable and isomorphic to its own square $X \oplus X$ then for $X$, (HBEP) is equivalent to local convexity.

In this section we will discuss this problem and give some observations. For convenience we will discuss only quasi-normed (i.e. locally bounded) spaces. Let us say that a quasi-Banach space $X$ has dense-(HBEP) if there is a dense linear subspace $V$ of $X$ with (HBEP).

**Lemma 3.1.** Let $X$ be a quasi-Banach space with dense-(HBEP). If $X$ has a separating dual then $X$ is locally convex.

**Proof:** This essentially follows from the arguments in [4]. However let us indicate a brief proof when $X$ is separable. In this case, if $X$ is non-locally convex it contains a proper closed weakly dense (PCWD) subspace $W$, [6]. Then it follows easily from the (HBEP) for $V$ that $W \supset V$ and this is a contradiction. From this it is easy to show the general case.
since if $X$ is non-locally convex it contains a separable non-locally convex subspace $X_0$ so that $X_0 \cap V$ is dense in $X_0$.

For an arbitrary F-space $X$ let $X^*$ denote its dual and put $N = N_X = \{ x : x^*(x) = 0 \ \forall x^* \in X^* \}$.

**Lemma 3.2.** Let $X$ be a quasi-Banach space with dense-(HBEP). If $X$ contains a basic sequence then $X$ is locally convex.

**Proof:** Clearly $N \cap V = \{ 0 \}$. Pick any $u \in N$ and non-zero $w \in V$. By standard perturbation arguments, $X$ contains a bounded basic sequence $(x_n)$ so that $x_n \in n(u + w) + V$ for each $n$. Let us write $x_n = n(u + w) + v_n$ where $v_n \in V$. For some $m_0 \geq 1$ and all $m \geq m_0$, we have $u + w \notin \bigcap_n = [x_k : k \geq m]$. Let $Y_m$ be the (closed) linear span of $Z_m$ and $u + w$. Then $v_n \in Y_m \cap V$ for $n \geq m$. Let $x^* \in X^*$ be a linear functional which vanishes on $Y_m \cap V$. Then $x^*(v_n) = 0$ and hence $x^*(u + w - n^{-1}x_n) = 0$. Thus $x^*(u + w) = 0$ and so $x^*(w) = 0$. Hence $w$ is in the weak-closure of $Y_m \cap V$ and by (HBEP) for $V$ this means that $w$ is in $Y_m$ for every $m \geq m_0$. Now the vectors $u + w$ and $\{ x_k : k \geq m_0 \}$ form a basis for $Y_{m_0}$ and so it follows easily that $w$ is a multiple of $u + w$ and thus $u = 0$. Hence $N_X = 0$ and so by the preceding Lemma 3.1, $X$ is locally convex.

**Lemma 3.3.** Let $X$ be a quasi-Banach space with dense-(HBEP). Let $E$ be a closed subspace of $N_X$. Then $X/E$ has dense-(HBEP).

**Proof:** Let $V$ as usual be the dense subspace of $X$ with (HBEP). If $Q : X \to X/E$ is the quotient map then $Q(V)$ is a dense subspace of $X/E$ with (HBEP). In fact if $W$ is a subspace of $Q(V)$ and $f$ is a continuous linear functional on $W$ then $f \circ Q$ can be extended to a continuous linear functional $x^*$ on $X$. Now since $x^*$ vanishes on $E$ it factors to a continuous linear functional on $X/E$.

**Theorem 3.4.** Let $X$ be a quasi-Banach space and let $N = \{ x \in X ; x^*(x) = 0 \ \forall x^* \in X^* \}$. In order that $X$ has dense-(HBEP) it is necessary and sufficient that $X/N$ is infinite-dimensional and locally convex and $X$ has the property that whenever $L$ is a closed subspace such that $\dim (L + N)/N = \infty$ then $L \supset N$.

**Proof:** First assume $X$ has dense-(HBEP). Then $X/N$ has dense-(HBEP) and so is locally convex by the preceding lemmas. Now suppose $L$ is a closed subspace of $X$ with $\dim (L + N)/N = \infty$. We may assume $N \neq \{ 0 \}$ and so by Lemma 8, $X$ is minimal. Then $Y = X/(L \cap N)$ has dense-(HBEP). If we assume that $L \cap N$ is a proper subset of $N$ then $Y$ is minimal. Let $M = L/L \cap N \subset Y$; then $M \cap N_Y = \{ 0 \}$. However $M$ is minimal and thus the quotient map $Q : Y \to Y/N_Y$ is an isomorphism on $M$. Thus $M$ is locally convex and so $\dim M < \infty$. However $\dim M = \dim (L + N)/N$ and so we have a contradiction and must conclude that $L \cap N = N$. 

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Conversely suppose $X$ has the properties listed. Let $V$ be an algebraic complement of $N$. First notice that the closure of $V$ must include $N$ by the hypotheses and so $V$ is dense in $X$. Let $\phi$ be a continuous linear functional defined on a relatively closed subspace $V_0$ of $V$. We show that $\phi$ can be extended to $V$ (or $X$.) If dim $V_0 < \infty$ this is immediate. Otherwise let $L$ be the closure of $V_0$ in $X$. Clearly $L \supset N$, and $\phi$ extends continuously to $L$. We also must have ker $\phi \supset N$. Now by applying the Hahn-Banach theorem to $X/N$ we obtain the result.

Let us give a simple application extending Ribe’s result cited above (Corollary E of [11]).

**Corollary 3.5.** Let $X$ be a quasi-Banach space with dense-(HBEP). Suppose $X = E \oplus F$ where both $E$ and $F$ are infinite-dimensional. Then $X$ is locally convex.

**Proof:** Note first that $N_X = N_E \oplus N_F$. We may assume without loss of generality that dim $E/N_E = \infty$. In this case dim $(E + N_X)/N_X = \infty$ and so $N_X \subset E$ and $N_F = \{0\}$. Thus dim $F/N_F = \infty$ and the same reasoning yields $N_E = \{0\}$. Hence $X$ has a separating dual and the result follows from Lemma 3.1.

**Theorem 3.6.** Let $X$ be a non-locally convex quasi-Banach space with dense-(HBEP). Then $X$ is minimal and $N = N_X$ is quotient-minimal.

**Proof:** If $X$ has dense-(HBEP) and is non-locally convex then clearly by Lemma 8 $X$ is minimal. Thus $N$ is also minimal. Further if $L$ is a closed subspace of $N$ then $N/L$ is a closed subspace of $X/L$ and so is also minimal.

We now can make a few observations on this problem. Let us assume $X$ is a non-locally convex quasi-Banach space with dense-(HBEP) and suppose $X$ has no atomic subspaces (e.g. if there are no atomic quasi-Banach spaces). Then $N$ is finite-dimensional. We can then choose $L$ to a subspace of $N$ of codimension one and replace $X$ by $X/L$. Thus we can assume that $N$ has dimension one. We conclude that we are looking for a minimal quasi-Banach space $X$ with a subspace $N$ of dimension one so that $X/N$ is a Banach space. This is exactly Problem 6. In fact we can go further and argue conversely that a positive solution to Problem 6 does indeed give a negative solution to Problem 7.

**Theorem 3.7.** Let $X$ be an infinite-dimensional minimal quasi-Banach space with a one-dimensional subspace $F$ so that $X/F$ is locally convex. Then $X$ has dense-(HBEP).

**Proof:** Clearly $F = N_X$. Further if $L$ is an infinite-dimensional closed subspace of $X$ and $F \cap L = \{0\}$ then since $L$ is minimal the quotient map $Q : X \to X/F$ is an isomorphism on $L$. But then $L$ is locally convex and we have a contradiction. Thus $L \supset F$.

Thus if we hold that there are no atomic quasi-Banach spaces, Problems 6 and 7 are equivalent.
4. The weak Hahn-Banach extension problem.

In this section we will discuss another question related to the Hahn-Banach Extension Property. This question is not directly linked to the atomic space problem, but nevertheless it seems to the author to involve similar ideas. In [15] Shapiro proved that an F-space with a basis with (HBEP) is locally convex; on further examination he proved a little more:

**Theorem 4.1 (Shapiro).** Let $X$ be a non-locally convex F-space with a basis. Then there is a weakly closed subspace (actually, the closed linear span of a block basic sequence) $E$ and a continuous linear functional $f \in E^*$ which cannot be extended to a continuous linear functional on $X$.

Motivated by this, let us say that an F-space $X$ with separating dual has the weak-(HBEP) if whenever $E$ is a weakly closed subspace of $X$ and $f \in E^*$ then $f$ can be extended to a continuous linear functional on $X$.

**Problem 8.** Let $X$ be an F-space with weak-(HBEP). Is $X$ locally convex?

As before, we will specialize this problem to quasi-Banach spaces, primarily for convenience of exposition. Let us make some initial remarks. Under this hypothesis if $E$ is a weakly closed subspace of $X$ then the Mackey topology of $E$ (i.e. the finest locally convex topology with the same dual) is the same as the relativized Mackey topology of $X$. Thus the Banach envelope norm for $E$ is equivalent to the Banach envelope norm for $X$. Further the Banach envelope of $E$ is isomorphic to the closure of $E$ in the Banach envelope $X_c$ of $X$.

**Theorem 4.2.** Let $X$ be a quasi-Banach space with weak-(HBEP). Then for every infinite-dimensional weakly closed subspace $E$, the quotient $X/E$ is locally convex.

**Proof:** We will assume that $X$ is normed by a $p$-subadditive quasi-norm where $0 < p < 1$. Let $Q : X \to X/E$ be the quotient map. Let $\| \|_c$ denote the Banach envelope norm on either $X$ or $X/E$. Next let $\| \|_0$ be the lower semi-continuous regularization of the quotient norm $\| \|$ on $X/E$ with respect to $\| \|_c$. Thus $\| \xi \|_0 = \liminf_{\| \eta - \xi \|_c \to 0} \| \eta \|$.

We aim to show that there exists a constant $K$ so that $\| \xi \|_0 \leq K \| \xi \|_c$ for $\xi \in X/E$. Once this is achieved then the identity map $I : (X/E, \| \|) \to (X/E, \| \|_c)$ is surjective and a standard form of the Open Mapping theorem (e.g. Theorem 1.4 of [7]) gives that I is open. This will imply that $X/E$ is locally convex.

Let us therefore assume the contrary that $\sup_{\| \xi \|_c \leq 1} \| \xi \|_0 = \infty$ and produce a contradiction. We start with the observation that if $F$ is a finite-dimensional subspace of $X/E$ and $B$ is a compact subset of $F$ then since $\| \|_0$ is $\| \|_c$-continuous on $F$ and $\| \|_c$-lower-semi-continuous on $X/E$ there exists a $\delta > 0$ so that $\| \eta \|_c \leq \delta$ and $\xi \in B$ then $\| \xi + \eta \|_0 > \| \xi \|_0 - 1$. 

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We now choose a sequence of positive numbers \((\delta_n)_{n=1}^{\infty}\) and a sequence \((\xi_n)_{n=1}^{\infty}\) of elements of \(X/E\). To start the induction choose \(\delta_1 = 1/10\). Now suppose that \(\delta_k\) has been chosen for \(k \leq n\) and \(\xi_k\) has been chosen for \(k < n\). We then choose \(\xi_n\) so that \(\|\xi_n\|_c < \delta_n/2\) and \(\|\xi\|_0 > n^3\). Once this is done we let \(B\) be the set of all \(\sum_{k=1}^{n} a_k \xi_k\) where \(|a_k| \leq 1\) for \(1 \leq k \leq n\) and choose \(\delta_{n+1}\) so that \(\delta_{n+1} < \delta_n/2\) and if \(\|\eta\|_c \leq \delta_{n+1}\) and \(\xi \in B\) then \(\|\xi + \eta\|_0 > \|\xi\|_0 - 1\).

Now pick \(x_n \in X\) so that \(Qx_n = \xi_n\) and \(\|x_n\|_c < \delta_n/2\). (This is possible since \(\|\|\|_c\) on \(X/E\) is the quotient norm of \(\|\|_c\).) Since \(E\) is infinite-dimensional we may pick a sequence \((y_n)\) in \(E\) with \(\|y_n\|_c = 1\) so that \((y_n)\) is a basic sequence in the Banach envelope \(X_c\) of \(X\) with constant at most two, i.e. so that for any \((a_k)_{k=1}^{n}\) and any \(m < n\)

\[
\|\sum_{k=1}^{m} a_k y_k\|_c \leq 2 \sum_{k=1}^{n} a_k y_k \|_c.
\]

Now notice that from the construction \(\sum_{k=1}^{n} \|x_n\|_c < 1/5\) and so by standard perturbation results if \(z_n = x_n + y_n\) then \((z_n)\) is also basic in the envelope \(X_c\). We let \(Z\) be the weak closure of \(\{z_n\}\) i.e. the intersection of the closed linear span in \(X_c\) with \(X\). Let \(z_n^*\) be the biorthogonal functionals in \(Z^*\). Since \(\|z_n\|_c \geq 9/10\) for all \(n\) we must have that \((z_n^*)\) is bounded in \((Z, \|\|_c)^*\). Let \(C\) be a constant such that \(\|z_n^*(z)\| \leq C \|z\|_c\) for all \(z \in Z\).

Suppose \(z \in Z\) and \(\|z\|_c \leq C^{-1}\). Thus \(\|z_n^*(z)\| \leq 1\) for all \(n\). Now, for any \(n\),

\[
\|\sum_{k=n+1}^{\infty} z_n^*(z) \xi_k\|_c < \frac{1}{2} \sum_{k=n+1}^{\infty} \delta_k < \delta_{n+1}.
\]

Thus

\[
\|\sum_{k=1}^{n} z_n^*(z) \xi_k\|_0 < \|Qz\|_0 + 1.
\]

However this implies by the p-subadditivity of \(\|\|_0\),

\[
|z_n^*(z)||\xi_n||_0 < 2^{1/p}(|Qz||_0 + 1).
\]

By construction we deduce that

\[
|z_n^*(z)| \leq 2^{1/p} n^{-3}(|Qz||_0 + 1).
\]

By homogeneity we deduce that \(\sup_{n \in \mathbb{N}} n^3 |z_n^*(z)| < \infty\) for every \(z \in Z\). It follows that we can define a linear functional \(f\) on \(Z\) by \(f(z) = \sum_{n=1}^{\infty} n z_n^*(z)\) and from the Banach-Steinhaus theorem \(f\) is continuous on \(Z\) for the original quasi-norm topology. Hence \(f\) can be extended continuously to \(X\) and is also \(\|\|_c\) continuous. Since \(f(z_n) = n\) we deduce that \(\|z_n\|_c \to \infty\) which is a contradiction, as promised.

This theorem allows us to solve the problem under certain circumstances.
COROLLARY 4.3. Let $X$ be a quasi-Banach space with weak-(HBEP). The $X$ is locally convex if one of the following hypotheses holds:

1. $X$ contains an infinite-dimensional weakly closed locally convex subspace.
2. $X$ contains a weakly closed subspace with a basis.
3. $X$ contains two infinite-dimensional subspaces $E$ and $F$ so that $X = E \oplus F$.
4. The Banach envelope $X_c$ of $X$ contains an infinite-dimensional $B$-convex subspace.

PROOF: (1) In this case suppose $E$ is an infinite-dimensional weakly closed subspace which is locally convex. Suppose $Q$ is the quotient map. If $\|x_n\|_c \to 0$ then $\|Qx_n\|_c \to 0$ and so by Theorem 4.2, $\|Qx_n\| \to 0$. Thus there exist $e_n \in E$ so that $\|x_n - e_n\| \to 0$. Hence $\|e_n\|_c \to 0$. However the relativized $\| \|_c$ topology on $E$ coincides with the Mackey topology of $E$ by weak-(HBEP) and since $E$ is locally convex this means that $\|e_n\| \to 0$. Thus $\|x_n\| \to 0$.

(2) Clearly the subspace with a basis also has weak-(HBEP) and so by Shapiro’s theorem is locally convex. This reduces us to (1).

(3) $E$ and $F$ are both also weakly closed and $E \sim X/F$ and $F \sim X/E$ are therefore both locally convex.

(4) In this case we may pick a basic sequence $(x_n)$ for $X_c$ with $x_n \in X$ and so that the closed linear span in $X_c$ is $B$-convex. Let $E$ be the weakly closed linear span in $X$. By the weak-(HBEP) $E_c$ is isomorphic to $[x_n]$ in $X_c$. Thus $E_c$ is $B$-convex and so $E$ is locally convex [5]. Hence by (1) $X$ is locally convex.

Let us mention a variant of Problem 8 which may be of interest to specialists in the theory of locally convex spaces.

PROBLEM 9. Let $X$ be a normed space which is fully barreled (i.e. every closed subspace is barreled). Is $X$ necessarily ultrabarreled?

An examination of Theorem 4.2 shows that this is what is required to improve the argument to solve Problem 8.

References.
ISOMETRIES OF ORLICZ SPACES

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The purpose of this paper is to give a brief historical review of the results about isometries of Orlicz spaces, some of their generalizations and some technics applied in this theory.

A linear operator $U$ of a Banach space $(X, \| \cdot \|)$ is called an isometry of $X$ if $\| Ux \| = \| x \|$ for all $x \in X$. We will say that $U$ is a surjective or invertible isometry if $U$ is an onto isometry of $X$. In what follows, $(T, \Sigma, \mu)$ will be a measure space where $\mu$ is a $\sigma$-finite measure defined on the $\sigma$-algebra $\Sigma$ of subsets of $T$. Let $\varphi$ be a Young function, that is $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(0) = 0$, $\varphi$ is convex and strictly increasing. Throughout the paper, suppose that $\varphi(1) = 1$. The Orlicz space $L_\varphi$ is then defined as the set of all complex or extended real valued functions defined on $T$ and such that $I_\varphi(\lambda f) \equiv \int \varphi(\lambda | f |) d\mu < \infty$. The space $L_\varphi$ is usually equipped with two standard norms, the Orlicz norm or the Luxemburg norm. In this paper we will investigate the space $L_\varphi$ with the Luxemburg norm defined as $\| f \| = \inf \{ \varepsilon > 0 : I_\varphi(L_\varepsilon) \leq 1 \}$. For $\varphi(u) = u^p, 1 \leq p < \infty$, $L_\varphi$ is the Lebesgue space $L^p$. A standard growth condition imposed on $\varphi$ is called a $\Delta_2$-condition and varies dependently on a measure space. Here we limit ourselves to say that, $\varphi$ satisfies $\Delta_2$-condition if $\sup_{u>0} \frac{\varphi(2u)}{\varphi(u)} < \infty$, and $\delta_2$-condition if $\sup_{0<u<1} \frac{\varphi(2u)}{\varphi(u)} < \infty$. Under the requirement of the $\Delta_2$-condition, $L_\varphi$ coincides with the closure of simple functions. For more information on Orlicz spaces see [15, 16, 20].

The Lebesgue or Orlicz spaces are particular examples of ideal Banach spaces called also Köthe or Banach function spaces. A Banach space $(E, \| \cdot \|)$ of measurable complex or real valued functions is called an ideal if the condition $| f(t) | \leq | g(t) |$ a.e., $g \in E$, implies $f \in E$ and $\| f \| \leq \| g \|$ ([18, 26, 27]).

A set map $\tau : \Sigma \to \Sigma$, defined modulo null sets, is called a regular set homomorphism ([17, 24]) if it satisfies the following conditions

(i) $\tau(A^c) = \tau(A)^c$, where $A^c$ is a complement of $A$, 

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(ii) \( \tau(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} \tau(A_i) \) for any sequence \( \{A_n\} \) of pairwise disjoint sets,

(iii) \( \mu \tau(A) = 0 \) if and only if \( \mu A = 0 \).

Such a set map induces a unique linear transformation of the class of \( \Sigma \)-measurable functions, characterized by \( \chi_A \mapsto \chi_{\tau A} \). We will denote this transformation as \( f \circ \tau \) for any measurable function \( f \); in particular \( \chi_A \circ \tau = \chi_{\tau A} \). We say that a set map \( \tau \) is induced by a point mapping \( \theta : T \to T \) if \( \tau A = \theta^{-1}[A] \), where \( \theta^{-1}[A] \) is an inverse image of \( A \) by \( \theta \). Not all set maps are induced by point mappings. However, if \( T \) is an uncountable, separable metric space with \( \sigma \)-algebra of Borel sets, then each regular set homomorphism \( \tau \) is induced by a point, Borel measurable and onto mapping \( \theta : T \to T \)[22].

**Function spaces**

Throughout this section suppose the measure \( \mu \) is atomless. Starting point in studying of isometries in spaces of integrable type, is a characterization of onto isometries of \( L^p \), \( 1 \leq p < \infty \), given by S. Banach in his classic monograph[2], in 1932.

The key point of the Banach's proof is to show the disjoint support property for isometries of \( L^p \) if \( p \neq 2 \).

It is said that an isometry \( U \) of an ideal Banach space \( E \), has the disjoint support property if \( f_1 f_2 = 0 \) a.e. implies that \( U f_1 U f_2 = 0 \) a.e.

**Fact 1.** If \( U \) is an isometry of any ideal Banach space \( E \) with the disjoint support property and simple functions dense in it, then there exist a measurable function \( w \) and a regular set homomorphism \( \tau \) of \( \Sigma \) such that

\[
U f = w \circ \tau
\]

for all \( f \in E \).

To show this, let \( \tau A = \operatorname{supp} U \chi_A \) whenever \( \chi_A \in E \). Let \( \{T_n\} \) be a partition of \( T \) such that \( \chi_{T_n} \in E \). By the disjoint support property, the function \( w = \sum_{n=1}^{\infty} U(\chi_{T_n}) \) is well defined and the transformation \( \tau \) is a regular set homomorphism. Thus the equation is obvious for simple functions, so for any function in \( E \).

S. Banach showed that an isometry \( U \) of \( L^p \), if \( 1 \leq p < \infty, p \neq 2 \), where \( L^p \) is the space of real valued, Lebesgue measurable functions on \([0,1]\), has the disjoint support property. Thus the isometry \( U \) of \( L^p, p \neq 2 \), has the form (**), where \( \tau \) is induced by a point mapping.
of \([0,1]\) onto itself. The idea of the proof is the following. Let \(p > 2\) and let \(g_i = U f_i\), where \(f_i \in L^p\), \(i = 1,2\), are arbitrary functions with disjoint supports. Then for any \(\alpha, \beta \in \mathbb{R}\),

\[
\int_0^1 |\alpha g_1 + \beta g_2|^p = |\alpha|^p \int_0^1 |g_1|^p + |\beta|^p \int_0^1 |g_2|^p.
\]

Computing the derivative with respect to \(\alpha\) in both sides of the above equality we get

\[
\int_0^1 p|\alpha g_1 + \beta g_2|^{p-1} \text{sgn} (\alpha g_1 + \beta g_2) g_1 = p \text{sgn} \alpha |\alpha|^{p-1} \int_0^1 |g_1|^p.
\]

Taking the second derivative we have

\[
\int_0^1 |\alpha g_1 + \beta g_2|^{p-2} |g_1|^2 = |\alpha|^{p-2} \int_0^1 |g_1|^p.
\]

Now setting \(\alpha = 0\) and \(\beta = 1\), \(\int_0^1 |g_2|^{p-2} |g_1|^2 = 0\) which yields \(g_1 g_2 = U f_1 U f_2 = 0\) a.e. If \(p < 2\), then the adjoint \(U^*\) has the disjoint support property, which implies that \(U\) has also this property.

In 1958, J. Lamperti [17] provided a different proof of the disjoint support property of an isometry of \(L^p\), based on the Clarkson's inequalities. In fact, he generalized the Clarkson's inequalities to some class of Young functions, and obtained a characterization of isometries of Orlicz spaces preserving the modular, i.e. \(I_\varphi(f) = I_\varphi(U f)\) for all \(f \in L_\varphi\). By the definition of the norm in \(L_\varphi\), the last equality implies that \(\| f \| = \| U f \|\). If we call a linear operator preserving a modular, a modular isometry, then obviously any modular isometry of \(L_\varphi\) is an isometry. We do not know if the converse is true in general, but as we will see later it is true for the spaces of complex valued functions.

**Lemma.** ([17] a generalization of the Clarkson's inequalities) If \(\varphi\) is a Young function such that \(\varphi(\sqrt{u})\) is a convex function of \(u\), then

\[
\varphi(|u + w|) + \varphi(|u - w|) \geq 2\varphi(|u|) + 2\varphi(|w|),
\]

for any \(u, w \in \mathbb{C}\), while if \(\varphi(\sqrt{u})\) is concave, the reverse inequality is true. Provided the convexity or concavity is strict, equality holds iff \(uw = 0\).

Now, the characterization of isometries given by Lamperti is the following.
Theorem 1. ([17]) Let $\varphi$ satisfy a $\Delta_2$ - condition.

I. Suppose $\varphi(\sqrt{u})$ is either strictly convex or strictly concave, and that $U$ is a modular isometry of $L_\varphi$, where $L_\varphi$ is the space of real or complex valued functions. Then there exist a regular set homomorphism $\tau$ and a measurable function $w$ such that

(i) \[ Uf = w \circ \tau \]

and

(ii) \[ \int_{\tau A} \varphi(|w|) = \int_{\tau A} \frac{d\mu \circ \tau^{-1}}{d\mu} \]

for all $A \in \Sigma$, and

(iii) \[ |w(t)| \in M_\varphi \text{ a.e.,} \]

where $M_\varphi$ is the set of multipliers of $\varphi$ that is $M_\varphi = \{ a > 0 : \varphi(au) = \varphi(a)\varphi(u) \text{ for all } u \geq 0 \}$.

Conversely, if $\tau$ is a regular set homomorphism and $w$ a measurable function satisfying (ii) and (iii), then $U$ given by (i) is a modular isometry of $L_\varphi$.

II. If $U$ is a surjective isometry of $L_\varphi$, then we obtain an analogous version of I replacing condition (ii) by

(ii') \[ \varphi(|w|) = \frac{d\mu \circ \tau^{-1}}{d\mu} \]

where $w(t) \neq 0$ a.e.

In fact the above statement is a corrected version of the Lamperti's result. He was not fully right claiming that (i), (ii') and (iii) are necessary conditions for $U$ to be an isometry (not necessarily surjective). This was pointed out by R. Grzaslewicz [12], who provided a counterexample.

The disjoint support property is then shown by J. Lamperti as follows. If $f_1 f_2 = 0$ a.e., then $I_\varphi(f_1 f_2) + I_\varphi(f_1 - f_2) = 2I_\varphi(f_1) + 2I_\varphi(f_2)$ and since $U$ is a modular isometry it implies that $I_\varphi(Uf_1 + Uf_2) + I_\varphi(Uf_1 - Uf_2) = 2I_\varphi(Uf_1) + 2I_\varphi(Uf_2)$. Thus applying the preceding lemma, $Uf_1 Uf_2 = 0$ a.e.

An attempt to remove the assumption of preserving the modular by an isometry in the Lamperti's theorem yields the following partial result. Let $\varphi'$ and $\varphi''$ denote the first and the second derivative of $\varphi$. 
Proposition 2. Assume that \( \varphi \) and \( \varphi' \) are strictly convex, \( \varphi'(0) = \varphi''(0) = 0 \) and \( \varphi \) satisfies a \( \Delta_2 \) - condition. If \( U \) is an isometry of the space \( L_\varphi \) of real valued functions, then \( U \) has the disjoint support property. Thus \( U \) is of the form \((*)\).

Proof. In the proof, Banach's method is applied. Setting \( g_i = U f_i, i = 1, 2, \) where \( f_i = \chi_{A_i} \) and \( A_1 \cap A_2 = \emptyset \), denote for any reals \( \alpha, \beta, f(\alpha, \beta) = \| \alpha \chi_{A_1} + \beta \chi_{A_2} \| = \| \alpha g_1 + \beta g_2 \| \). Then \( \| \frac{\alpha g_1 + \beta g_2}{f(\alpha, \beta)} \| = 1 \), which implies that \( G(\alpha) = I_\varphi(\frac{\alpha g_1 + \beta g_2}{f(\alpha, \beta)}) = 1 \), by the assumption of \( \Delta_2 \) - condition. The assumptions imposed on \( \varphi \) allows us to differentiate \( G \) twice with respect to \( \alpha \). Then setting \( \alpha = 0 \) and \( \beta = 1 \), \( \int \varphi''(\| g_2 \|)g_1g_2 = 0 \). Hence \( g_1 g_2 = 0 \) a.e.

In 1963, G. Lumer[19] characterized surjective isometries of reflexive Orlicz spaces of complex valued functions. His method employing the theory of Hermitian operators has become a fruitful tool in studying isometries of complex Banach spaces; the method is applicable to complex spaces. In 1976, M.G. Zaidenberg[26] extended the Lumer's result, removing the assumption of reflexivity and separability of the space. Before stating the main result and the outline of its proof, we provide some definitions and auxiliary facts.

A bounded linear operator \( H \) of a complex Banach space \( X \) is said to be Hermitian if \( x^* (Hx) \) is real for every pair of vectors \( x \in X, x^* \in X^* \) such that \( x^* \) is a support functional of \( X \) i.e. \( \| x \| = \| x^* \| \) and \( x^*(x) = \| x \|^2 \). Equivalently, \( H \) is Hermitian iff \( \| e^{i\alpha}H \| = 1 \) for every \( \alpha \in \mathbb{R} \) or \( e^{i\alpha}H \) is a surjective isometry of \( X \) for each \( \alpha \in \mathbb{R}[3] \).

Fact 2. (14) If \( H \) is a Hermitian operator of a Banach space \( X \) and \( U \) a surjective isometry of \( X \), then \( UHU^{-1} \) is a Hermitian operator.

Fact 3. (26) If \( E \) is an ideal Banach space and \( U \) a surjective isometry of \( E \) such that for any \( \chi_A \in E, U \chi_A = w \chi_{\tau A} \), where \( w \) is a measurable function and \( \tau \) a regular set homomorphism, then \( Uf = w f \circ \tau \) for all \( f \in E \).

The next result, a characterization of Hermitian operators, plays a fundamental role in the characterization of isometries of \( L_\varphi \).

Theorem 3. (19, 26) Let \( \varphi(u) \neq cu^2 \). Then any bounded linear operator \( H \) of a complex space \( L_\varphi \) is Hermitian if and only if \( Hf = hf \) for all \( f \in L_\varphi \), where \( H \) is a real bounded function; moreover \( \| H \| = \| h \|_\infty \).

Comments to the proof. The sufficiency part was given by G. Lumer in [19] as Lemma 7. He provides a proof, which works also for ideal Banach function spaces. For the necessity, the essential part is to show that \( \text{supp} \, H \chi_A \subseteq A \). The proof of that is based on the known
representation of the support functional in $L_\varphi$ and some special lemmas (cp. Lemma 5 in [19], Lemma 1.3 in [13]). Hence there exists a real and bounded function $h$, such that $H_{\chi_A} = h_{\chi_A}$ (in particular, if $\mu T < \infty$, then $h = H1$). Now, if the simple functions are dense in $L_\varphi$, that is if $\varphi$ satisfies a $\Delta_2$-condition, then $Hf = hf$ obviously. Zaidenberg removed the assumption of the $\Delta_2$-condition in the following way. Since $H_{\chi_A} = h_{\chi_A}$ and $H$ is Hermitian, $e^{iaH}\chi_A = e^{iah}\chi_A$ and $e^{iaH}$ is an isometry. Hence $e^{iaH}f = e^{iah}f$ for all $f \in L_\varphi$, by Fact 3. Now, by the differentiation of the group of isometries we get that $Hf = hf$ for all $f \in L_\varphi$.

**Theorem 4.** ([19, 26]) Let $\varphi(u) \neq cu^2$. If $U$ is a surjective isometry of a complex space $L_\varphi$, then there exist a measurable function $w$ and a regular set homomorphism $\tau$ of $\Sigma$ such that

(*)

$Uf = w f \circ \tau$

for every $f \in L_\varphi$, and

(**)

$\varphi(|w(t)|, \lambda) = \tau'(t)\varphi(\lambda)$

for all $\lambda \geq 0$, a.a. $t \in T$, where

$\tau' = \frac{d\mu \circ \tau^{-1}}{d\mu}$.

Conversely, if a set homomorphism $\tau$ and a function $w$ are related by (**), then $U$ given by (*) is an isometry. Moreover, any surjective isometry of $L_\varphi$ is also a modular isometry.

**Comments to the proof.** Direct calculations show that conditions (*) and (**) imply the equation $L_\varphi(f) = L_\varphi(Uf)$, which proves that $U$ is a modular isometry. This part of the proof is not dependent on whether $L_\varphi$ is real or complex space. To prove necessity of the condition (*), the previous result about Hermitian operators is employed. We will sketch this part under the assumption that $\mu T < \infty$. For $A \in \Sigma$, $H_A^2 = f_{\chi_A}$ is Hermitian. Since $U$ is a surjective isometry, $UH_AU^{-1}$ is also Hermitian, by Fact 2. Hence $UH_AU^{-1}f = H_Af$ for a real and bounded function $h_A$. Now since $UH_AU^{-1}$ is a projection, $h_A^2 = h_A$. This implies that $h_A(t) = 0$ or $h_A(t) = 1$. Setting $\tau A = \{t \in T : h_A(t) = 1\}$, $\tau$ is a regular set homomorphism and $h_A = \chi_{\tau A}$. Thus $U_{\chi A} = UH_AU^{-1}U1 = (U1)_{\chi_{\tau A}} = (U1)_{\chi A} \circ \tau$. Now we get (*) by Fact 2. If we have already equation (*), then equation (**) is satisfied either for real or complex space (see proofs of Theorem 3 in [26] or Theorem 8 in [10]).
PROBLEM. Is Theorem 4 also true for real Orlicz space?

Partial answers to this problem are contained in Theorem 1 and Proposition 2. Conditions (i), (ii') and (iii) of Theorem 1 and (⋆), (⋆⋆) of Theorem 4 are equivalent. The question is, can we prove Theorem 1 or Proposition 2 without additional assumptions imposed on φ?

J. Lamperti (Lemma 4 in [17]) showed that the group of multipliers $M_\varphi$ has the following property: either $M_\varphi = \mathbb{R}_+$ and then $\varphi(u) = u^p$ for some $p$; or $M_\varphi$ is an infinite cyclic group with a generator $k \neq 1$ or $M_\varphi = \{1\}$. This and the previous theorem yields

**Theorem 5.** ([26]) Let $L_\varphi$ be a complex Orlicz space. We have the following classification of surjective isometries of $L_\varphi$.

(a) $M_\varphi = \{1\}$. Then any surjective isometry of $L_\varphi$ is of the form (⋆), where $\tau$ is a measure preserving homomorphism i.e. $\mu \tau^{-1} A = \mu A$, and $|w(t)| = 1$.

(b) $M_\varphi$ is an infinite cyclic group with a generator $k$. Then any surjective isometry of $L_\varphi$ is of the form (⋆), where $|w(t)| \in M_\varphi$ a.e. and $\varphi(|w(t)|) = \tau^r(t)$.

(c) $M_\varphi = \mathbb{R}_+$ and $\varphi(u) = u^p$, $p \neq 2$. Then $L_\varphi = L^p$ and any surjective isometry is of the form (⋆), where $|w(t)|^p = \tau^r(t)$.

(d) $M_\varphi = \mathbb{R}_+$ and $\varphi(u) = u^2$. Then $L_\varphi = L^2$, and any surjective isometry is an unitary operator.

**REMARK.** If $(T, \Sigma, \mu)$ is the Lebesgue measure space, then a set homomorphism $\tau$ may be replaced by a point mapping.

Now let me mention some generalizations and extensions of Theorem 4. M.G. Zaidenberg [27] proved that a surjective isometry of a symmetric Banach space of complex valued functions has the form (⋆). He also gave a classification of the group of isometries in these spaces. The isometries of Musielak - Orlicz spaces, which are not symmetric in general, were described in [10, 13]. In particular, we get a representation of isometries of Nakano spaces. Let $p(t)$ be a measurable function with $1 \leq p(t) < \infty$. Then the Nakano space $L^{p(t)}$ is the set of all measurable functions $f$ such that $\int \frac{|f(t)|^{p(t)}}{p(t)} < \infty$ for some $\lambda > 0$. This is a Banach space equipped with the norm $\|f\| = \inf \{\epsilon > 0 : \int \frac{1}{p(t)} |f(t)|^{p(t)} \leq 1\}$.

**Theorem 6.** ([10]) Let $L^{p(t)}$ be a Nakano space of complex valued functions, with $p(t) \neq 2$. If $U$ is a surjective isometry of $L^{p(t)}$, then there exist a measurable function $w$ and a regular set homomorphism $\tau$ such that

$$Uf = w(f \circ \tau)$$
for \( f \in L^p(t) \), and

\[
(p(t) = (p \circ \tau)(t), \quad w(t) |p(t) = \frac{d\mu \circ \tau^{-1}}{d\mu}(t).
\]

Conversely, if \( w \) and \( \tau \) satify (2), then \( U \) given by (1) is an isometry.

A partial result of the above has been shown also for real Nakano spaces [13]. A.R.Sourour [24] and P.Greim [11] characterized isometries of Lebesgue - Bochner spaces \( L^p(X) \). This result was partially generalized to Orlicz - Bochner spaces, by J.Jamison and I.Loomis in [14]. Positive isometries of \( L_\varphi \) were studied in [5]. Isometries of Orlicz spaces generated by a concave function \( \varphi \), were discussed in [6, 9].

**SEQUENCE SPACES**

In this section let \( \mu \) be a purely atomic measure. Let further \( l_\varphi \) denote the Orlicz space, if the measure \( \mu \) of each atom is equal to one, and let \( l_\varphi(\{a_n\}) \) denote Orlicz space if the masses of atoms are equal to \( a_n \). Here we will discuss only infinite dimensional Orlicz spaces. Thus the elements of \( l_\varphi \) or \( l_\varphi(\{a_n\}) \) are infinite sequences of real or complex numbers. The space \( l_\varphi \) is separable and has 1 - symmetric basis (composed of unit vectors) iff \( \varphi \) satisfies a \( \delta_2 \) - condition [18]. This basis is also hyperorthogonal [23].

S.Banach, in his monograph, showed that a linear isometry on a real sequence space \( l^p \), where \( 1 \leq p < \infty, p \neq 2 \), has the disjoint support property. Then he gave the following representation of onto isometries of \( l^p, p \neq 2 \).

**Theorem 7.** ([2,18]) For any surjective isometry \( U \) of \( l^p \), where \( p \neq 2 \), there exist a permutation \( \pi \) of \( \mathbb{N} \) and a sequence \( \varepsilon_n = \pm 1 \), such that

\[
(Ux)_n = \varepsilon_n x_{\pi(n)}
\]

for every \( x = (x_n) \in l^p \). Conversely, any operator \( U \) of the form (\( + \)) is an onto isometry of \( l^p \).

Since then, this theorem has been generalized to many other spaces. As in the case of function spaces, it is necessary to treat real and complex spaces separately. The most general result in the case of real spaces belongs to M.S.Braverman and E.M.Semenov [4] (see also [21]). Let \( X \) be a Banach space, in which elements are infinite sequences of real numbers, and let \( X \) be symmetric in the sense that for any permutation \( \pi \) of \( \mathbb{N} \) if
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\[ x = (x_n) \in X \text{ then } (x_{\pi(n)}) \in X \text{ and } \|x\| = \| (x_{\pi(n)}) \|. \] They proved that if \( X \not\equiv l^2 \) (\( X \equiv l^2 \) means that \( X \) and \( l^2 \) contain the same elements and norms are proportional), then any onto isometry of \( X \) has the form \((+)\). We have that \( l_\varphi \equiv l^2 \) iff \( \varphi(u) = u^2 \) for \( u \in [0,1] \) (e.g. [7]). Thus as a corollary of Braverman’s and Semenov’s result we get the following

**Theorem 8.** Let \( \varphi(u) \neq u^2 \) for \( u \in [0,1] \) and \( l_\varphi \) be a real sequence Orlicz space. A linear operator \( U \) is a surjective isometry of \( l_\varphi \) if and only if \( U \) is of the form \((+)\).

We have also results in the case of complex sequence spaces. Hermitian operators are employed to characterize isometries in this setting also. In [1], J.Arazy showed that if \( X \) is a complex Banach space with a \( 1 \) - symmetric basis, then assuming that \( X \not\equiv l^2 \), a linear operator \( U \) is a surjective isometry of \( X \) iff there exists a permutation \( \pi \) of \( \mathbb{N} \) and a unimodular sequence \( \{\lambda_k\} \) (i.e. \( |\lambda_k| = 1 \)) such that

\[(++)\]

\[(Ux)_n = \lambda_n x_{\pi(n)}\]

for all \( x = (x_n) \in X \). A similar result was obtained earlier by K.W.Tam in [25], with an additional assumption on \( X \). Namely, he assumed that the basis of \( X \) was \( 1 \) - symmetric and hyperorthogonal , so \( X \) was a lattice. Applying either Tam’s or Arazy’s result to \( l_\varphi \) we get the following characterization of isometries.

**Theorem 9.** Let \( \varphi(u) \neq u^2 \) for \( u \in [0,1] \) and \( \varphi \) satisfy a \( \delta_2 \) - condition. Let \( l_\varphi \) be a complex sequence Orlicz space. Then \( U \) is a surjective isometry of \( l_\varphi \) if and only if \( U \) is of the form \((++)\).

It seems to be possible to omit the assumption of the condition \( \delta_2 \) in the above theorem, following the Zaidenberg’s method for function spaces (cp. Fact 3).

R.Fleming and J.Jamison generalized Tam’s theorem, avoiding the assumption of symmetry of the space. They characterized Hermitian operators in some class of not necessarily symmetric sequence spaces [8] and then applied it to find a description of isometries of \( l_\varphi(\{a_n\}) \) [7]. Recall that the space \( l_\varphi(\{a_n\}) \) is separable iff \( \varphi \) satisfies a generalized \( \delta_2 \) - condition or iff the set of unit vectors forms an unconditional basis [18]. Here we provide a reformulated version of the Fleming’s and Jamison’s result.

**Theorem 10.** Let \( \varphi(u) \neq cu^2 \) for \( u \in [0,\max\{\frac{1}{\sqrt{cd}}, \frac{1}{\sqrt{cd}}\}] \), any \( c > 0 \) and any pair of natural numbers \( k,j \). Let the complex sequence space \( l_\varphi(\{a_n\}) \) be separable. If \( U \) is a surjective isometry of \( l_\varphi(\{a_n\}) \), then there exist a sequence \( \{\lambda_n\} \) and a permutation \( \pi \) of \( \mathbb{N} \) such that \( (Ux)_n = \lambda_n x_{\pi(n)} \).
References


ON ORLICZ RESULTS IN INTERPOLATION THEORY

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Dedicated to the memory of Professor
Władysław Orlicz (1903-1990)

Four Orlicz results in interpolation theory are presented with some remarks and commentaries.

0. INTRODUCTION

In 1935 Orlicz proved interpolation property of any separable Orlicz space \( L_\varphi(I) \) between \( L_1(I) \) and \( L_\infty(I) \), \( I = [0,1] \), for linear operators. Next, in 1954 he extended this theorem to arbitrary Orlicz spaces \( L_\varphi(I) \) and more generally to Lipschitz operators. These theorems had great influence on later investigations. In his third paper, from 1954, is proved interpolation property of \( H^\omega(I) \) between \( C(I) \) and \( H^1(I) \) for linear operators. This is the first interpolation theorem going beyond the scope of solid spaces. Unfortunately this theorem is unknown in the literature. Many years later O'Neil, Peetre and Semenov carried out analogous investigations. Let us mention that Orlicz proved his theorem even in the vector-valued case, i.e., that \( H^\omega(I,X) \) is an interpolation space between \( C(I,X) \) and \( H^1(I,X) \) for linear operators. Then, in 1980, he gave a generalization of the Riesz-Thorin theorem, on Musielak-Orlicz sequence spaces.

In this paper we will give some historical remarks, new proofs and commentaries about these theorems. Open problems are also given. Paper is divided into four parts. Each part contains remarks and commentaries about each of his result. I should mention that today there is a large amount of literature about interpolation of operators (see bibliography [20] with a more than two thousand papers).
1. INTERPOLATION OF LINEAR OPERATORS BETWEEN $L_1$ AND $L_\infty$

W. Orlicz proved in 1935 that any separable Orlicz space $L_\varphi(I)$ is an interpolation space between $L_1(I)$ and $L_\infty(I)$, $I = [0, 1]$, i.e., if $T$ is any linear bounded operator in $L_1(I)$ with the norm $M_1$ which is also bounded in $L_\infty(I)$ with the norm $M_\infty$, Then $T$ is bounded in $L_\varphi(I)$ with the norm $M \leq C \max(M_\infty, M_1)$, $C \geq 1$. His proof contains three steps:

1. the proof that integral operators which are bounded in $L_1(I)$ and in $L_\infty(I)$ are also bounded in $L_\varphi(I)$,
2. if such a type of operators are convergent on a dense subset of $L_\varphi(I)$ then they are convergent on $L_\varphi(I)$,
3. the expansion of any $x \in L_1(I)$ into the Haar-Fourier series and properties of this expansion.

This theorem was generalized on other spaces: Lorentz [12] proved it for integral operators and rearrangement-invariant spaces, Mitjagin [23] for arbitrary operators in separable symmetric spaces or symmetric spaces dual to separable spaces, Calderón [5] described all interpolation spaces between $L_1$ and $L_\infty$.

A Banach space $E$ of measurable functions (classes) on $I$ is called symmetric if

(i) $|x| \leq |y|$ a.e. and $y \in E$ imply $\|x\|_E \leq \|y\|_E$,
(ii) equimeasurability of $|x|$ and $|y|$ imply $\|x\|_E = \|y\|_E$.

In many publications devoted to symmetric spaces the Fatou property of the norm is additionally required and such spaces are so-called rearrangement-invariant spaces. Of course, all results in this part are also valid for functions defined on $I = [0, \infty)$ and also for spaces of sequences for which axioms (i) and (ii) are satisfied.

**THEOREM 1 (Calderón-Mitjagin).** Let $E$ be an intermediate space for $\{L_1, L_\infty\}$, i.e., $L_1 \cap L_\infty \subset E \subset L_1 + L_\infty$. The space $E$ is an interpolation space between $L_1$ and $L_\infty$ if and only if the following condition holds: there exists $C \geq 1$ such that

(1) $\frac{1}{t} \int_0^t x^*(s) ds \leq \int_0^t y^*(s) ds \quad \forall t > 0$, then $x \in E$ and $\|x\|_E \leq C \|y\|_E$.

It can be shown that for a separable symmetric space or a symmetric space with the Fatou property of the norm (=rearrangement-invariant space) the condition (1) is satisfied with $C=1$. It was also natural to ask: does any symmetric space satisfy condition (1)? G.I. Russu [35] constructed an example of a space without property (1). A more general result is contained in [10]. The question turned out to be closely connected with the boundedness of averaging operators (see Mekler [22]).

Since the Peetre $K$-functional for the pair $\{L_1, L_\infty\}$ has exact representation

(2) $K(t, x; L_1, L_\infty) = \int_0^t x^*(s) ds$,
the Calderón-Mitjagin theorem can be formulated in the following manner: an intermediate space \( E \) for \( \{L_1, L_{\infty}\} \) is an interpolation space between \( L_1 \) and \( L_{\infty} \) if and only if \( K(t, x; L_1, L_{\infty}) \leq K(t, y; L_1, L_{\infty}) \) for all \( t > 0 \) and \( y \in E \) implies that \( x \in E \) and \( \|x\|_E \leq C\|y\|_E \).

Pairs of Banach spaces \( \{X_0, X_1\} \) possessing an analogous property with \( K(t, ; X_0, X_1) \) instead of \( K(t, ; L_1, L_{\infty}) \) are called the Calderón pairs (C-pairs) or \( K \)-monotonic pairs. It turns out that this is equivalent to the following condition:

there exists a constant \( 0 < \lambda < \infty \) such that whenever \( K(t, y; X_0, X_1) \leq K(t, x; X_0, X_1) \) holds for all \( t > 0 \) then there exists a linear operator \( T \) bounded in \( X_0 \) and \( X_1 \) with the norms less than \( \lambda \) such that \( Tx = y \).

Calderón-Mitjagin theorem means that \( \{L_1, L_{\infty}\} \) is C-pair. Lorentz and Shimogaki [14] proved that \( \{L_1, L_p\} \) and \( \{L_q, L_{\infty}\} \) are C-pairs. Sparr [37] proved that \( \{L_{p_0}, L_{p_1}\}, 1 \leq p_0, p_1 \leq \infty \) is C-pair. On the other hand, \( \{\ell_1(L_1), \ell_1(L_{\infty})\} \) is not a C-pair (see [31]). Peetre conjectured that a pair of rearrangement-invariant spaces is a C-pair. However, Ovchinnikov [30] showed, that the pair \( \{L_1(I) + L_{\infty}(I), L_1(I) \cap L_{\infty}(I)\} \) on \( I = [0, \infty) \), is not a C-pair. More precisely, in [21] is proved that if \( I = [0, \infty) \) and \( p \neq 2 \) then \( L_{p}(I) \cap L_{p'}(I) \) and \( L_{p}(I) + L_{p'}(I) \) are interpolation spaces between \( \{L_1(I) + L_{\infty}(I), L_1(I) \cap L_{\infty}(I)\} \) but obtained not by the \( K \)-method.

For more details concerning the Calderón-Mitjagin theorem, see [1], [10]; for the Calderón pairs, see [7], [8] and [34].

Finally, let us formulate some (known) open problems.

**PROBLEM 1.** Does \( \{E, L_{\infty}\} \) is a Calderón pair for any rearrangement invariant space \( E \)?

**PROBLEM 2.** Prove or disprove the following conjecture of Yu. A. Brudnyi:

If for the pair of Orlicz spaces \( \{L_{\varphi}, L_{\varphi'}\} \) the lower and upper Orlicz-Matuszewska indices satisfy \( \alpha_{\varphi} < \beta_{\varphi} \) or/and \( \alpha_{\varphi_1} < \beta_{\varphi_1} \), then \( \{L_{\varphi}, L_{\varphi'}\} \) is not Calderón pair.

**PROBLEM 3.** Determine the necessary and sufficient conditions on \( \varphi_0 \) and \( \varphi_1 \), under which the pair \( \{L_{\varphi_0}, L_{\varphi_1}\} \) of Orlicz spaces is a Calderón pair.

2. **INTERPOLATION OF LIPSCHITZ OPERATORS BETWEEN \( L_1 \) AND \( L_{\infty} \)**

In 1954 Orlicz extended his own first theorem to arbitrary Orlicz spaces and more generally to Lipschitz operators. In this part we give a generalization of Orlicz's non-linear interpolation theorem with a simple proof. We did not succeed in proving the theorem for Orlicz space \( L_{\varphi}(\mu) \) but only for \( L_{\varphi}(\mu) \cap L_1(\mu) \). Simplicity of our proof stems from the application of the representation (6) of Orlicz functions and a special property (7) of the operators considered. In Orlicz's paper there was an additional constant connected with his technique of proof. This part is taken mainly from the paper [18] (cf. [19]).
THEOREM 2. Let \( T \) be an operator from \( L_1(\mu) + L_\infty(\mu) \) into \( L_1(\nu) + L_\infty(\nu) \). Assume that \( T \) is a Lipschitz operator from \( L_1(\mu) \) into \( L_1(\nu) \), i.e.,

\[
\|Tx - Ty\|_{L_1(\nu)} \leq M \|x - y\|_{L_1(\mu)} \quad \forall x, y \in L_1(\mu).
\]

(a) If

\[
\|Tx\|_{L_\infty(\nu)} \leq M \|x\|_{L_\infty(\mu)} \quad \forall x \in L_\infty(\mu),
\]

then \( T \) maps \( L_\varphi(\mu) \cap L_1(\mu) \) into \( L_\varphi(\nu) \) and

\[
\|Tx\|_{L_\varphi(\nu)} \leq M \|x\|_{L_\varphi(\mu)} \quad \forall x \in L_\varphi(\mu) \cap L_1(\mu).
\]

(b) If

\[
\|Tx - Ty\|_{L_\infty(\nu)} \leq M \|x - y\|_{L_\infty(\mu)} \quad \forall x, y \in L_\infty(\mu),
\]

then \( T \) maps \( L_\varphi(\mu) \cap L_1(\mu) \) into \( L_\varphi(\nu) \) and

\[
\|Tx - Ty\|_{L_\varphi(\nu)} \leq M \|x - y\|_{L_\varphi(\mu)} \quad \forall x, y \in L_\varphi(\mu) \cap L_1(\mu).
\]

Proof. (a) By taking \( T/M \) instead of \( T \) if necessary, we may assume that \( M = 1 \).
First, every convex Orlicz function \( \varphi \) has a representation

\[
\varphi(u) = au + \int_0^u (u - s)_+ dp(s),
\]

where \( p \) is the right-derivative of \( \varphi \) and \( a = p(0^+) \).

Indeed,

\[
\varphi(u) = \int_0^u p(s)ds = up(u) - \int_0^u sdp(s) = up(0^+) + \int_0^u udp(s) - \int_0^u sdp(s)
\]

\[= au + \int_0^u (u - s)dp(s) = au + \int_0^\infty (u - s)_+ dp(s). \]

Secondly, we prove that if (4') holds with \( M = 1 \), then for each \( x \in L_1(\mu) + L_\infty(\mu) \)

\[
|Tx(t) - (Tx)^{(\alpha)}(t)| \leq |Tx(t) - T(x^{(\alpha)})(t)| \quad \nu - a.e,
\]

where \( z^{(\alpha)}(t) = \min(|z(t)|, \alpha)sgn z(t), \alpha > 0 \).

Indeed, if \( |Tx(t)| \leq \alpha \) then, (7) is obvious. On the other hand, if \( |Tx(t)| > \alpha \), then since \( \|T(x^{(\alpha)})\|_\infty \leq \|x^{(\alpha)}\|_\infty \leq \alpha \), it follows that \( |T(x^{(\alpha)})(t)| \leq \alpha \quad \nu - a.e. \) Hence,

\[
|Tx(t) - (Tx)^{(\alpha)}(t)| = |Tx(t) - a\text{sgn}T(t) - (Tx)(t)| = |Tx(t) - \alpha|
\]

\[\leq |Tx(t)| - |T(x^{(\alpha)})(t)| \leq |Tx(t) - T(x^{(\alpha)})(t)| \quad \nu - a.e,
\]
Now, if \( x \in L_\varphi(\mu) \cap L_1(\mu) \) then from the representation (6) of \( \varphi \) and the Fubini theorem we have that

\[
I_\varphi(Tx/\lambda) = \int_{\Omega_1} \varphi(|Tx(t)|/\lambda)d\nu
\]

\[
= \int_{\Omega_1} [a|Tx(t)|/\lambda + \int_0^\infty (\frac{|Tx(t)|}{\lambda} - s)_+ dp(s)]d\nu
\]

\[
= \frac{a}{\lambda} ||Tx||_{L_1(\nu)} + \frac{1}{\lambda} \int_0^\infty \int_{\Omega_1} (|Tx(t)| - s\lambda)_+ dv dp(s)
\]

\[
= \frac{a}{\lambda} ||Tx||_{L_1(\nu)} + \frac{1}{\lambda} \int_0^\infty \int_{\Omega_1} |Tx(t) - (Tx)^{(s\lambda)}(t)| dv dp(s)
\]

\[
= \frac{a}{\lambda} ||Tx||_{L_1(\nu)} + \frac{1}{\lambda} \int_0^\infty ||Tx - (Tx)^{(s\lambda)}||_{L_1(\nu)} dp(s).
\]

Using (7) and (3) with \( M = 1 \) and, again, Fubini theorem and (6) we have

\[
I_\varphi(Tx/\lambda) \leq \frac{a}{\lambda} ||Tx||_{L_1(\nu)} + \frac{1}{\lambda} \int_0^\infty ||Tx - (Tx)^{(s\lambda)}||_{L_1(\nu)} dp(s)
\]

\[
\leq \frac{a}{\lambda} ||x||_{L_1(\mu)} + \frac{1}{\lambda} \int_0^\infty ||x - (x)^{(s\lambda)}||_{L_1(\mu)} dp(s) = I_\varphi(x/\lambda).
\]

Hence \( Tx \in L_\varphi(\nu) \) and

\[
||Tx||_{L_\varphi(\nu)} := \inf\{\lambda \geq 0 : I_\varphi(Tx/\lambda) \leq 1\} \leq ||x||_{L_\varphi(\mu)}.
\]

(b) For any fixed \( x_o \in L_1(\mu) \cap L_\infty(\mu) \) and for \( x \in L_1(\mu) + L_\infty(\mu) \), let

\[
Sx := T(x + x_o) - Tx_o.
\]

Then

\[
||Sx - Sy||_1 = ||T(x + x_o) - T(y + x_o)||_1 \leq M||x - y||_1 \ \forall x, y \in L_1(\mu)
\]

\[
||Sx||_\infty = ||T(x + x_o) - T(x_o)||_\infty \leq M||x||_\infty \ \forall x \in L_\infty(\mu).
\]

From (a) we get

\[
||Sx||_\varphi \leq M||x||_\varphi \ \forall x \in L_\varphi(\mu) \cap L_1(\mu).
\]

This means that

\[
||Tx - T_{x_o}||_\varphi \leq M||x - x_o||_\varphi \ \forall x \in L_\varphi(\mu) \cap L_1(\mu), \ \forall x_o \in L_1(\mu) \cap L_\infty(\mu).
\]

For arbitrary \( x, y \in L_\varphi(\mu) \cap L_1(\mu) \) we consider the truncations \( x^{(k)}, y^{(k)} \). Then \( z_k := T(x^{(k)}) - T(y^{(k)}) \) converges to \( Tx - Ty \) in the \( L_1(\nu) - \text{norm} \) by (3) and the continuity of
the $\| \cdot \|_1$-norm. Consequently, the same convergence holds in the measure $\nu$. Therefore, for a property chosen sequence $k_n$, the sequence $z_{k_n}$ converges $\nu$-$a.e.$ to $Tx - Ty$. Then $x_n := x^{k_n}$ and $y_n := y^{k_n}$ have the following property:

$$x_n, y_n \in L_1(\mu) \cap L_\infty(\mu), |x_n - y_n| \leq |x - y| \quad \mu$-a.e.$$

$$Tx_n - Ty_n \to Tx - Ty \quad \nu$-a.e.$$

(9)

Now, by (9), the Fatou property of the norm and (8) we get

$$\|Tx - Ty\|_\varphi \leq \liminf_{n \to \infty} \|Tx_n - Ty_n\|_\varphi$$

$$\leq M \liminf_{n \to \infty} \|x_n - y_n\|_\varphi \leq M \|x - y\|_\varphi.$$

**REMARKS**

1. Derivation of (b) from (a) has a general character and can be formulated in the language of abstract interpolation (cf. [33] and [17]).

2. If we assume that either $\mu \Omega < \infty$ or $a = p(0^+) > 0$ (then $L_\varphi \cap L_1 = L_\varphi$), or $T$ is linear, then our Theorem 2 holds even on $L_\varphi(\mu)$ space (cf. [18]).

3. Interpolation of Lipschitz operators between general Banach spaces was considered by Browder [3], Maligranda [17] and Brudnyi-Kruglak [4], but out Theorem 2 does not follow, in general, form these considerations. Some particular cases of Theorem 2 (for example, when $T$ additionally is positive or order preserving) were proved also by Crandall-Tartar [6] and Krzegel-Lin [11].

Applications of Theorem 2 to the proof of Brudnyi-Chiti, Hardy-Littlewood-Pólya, Lorentz-Shimogaki and Brudnyi-Oswald-Wik inequalities are given in [18] and [19].

Orlicz theorem (for Lipschitz operators) and Calderón-Mitjagin theorem (for linear operators) were extended to symmetric spaces and to Lipschitz operators by Lorentz-Shimogaki[13] and Maligranda [15]. Roughly speaking, in these papers, it is proved that if $E$ is an interpolation space between $L_1$ and $L_\infty$ for linear operators, then $E \cap L_1$ is an interpolation space between $L_1$ and $L_\infty$ for Lipschitz operators.

**PROBLEM 4.** Prove (or disprove) Theorem 2 on the whole space $L_\varphi(\mu)$, not only on $L_\varphi(\mu) \cap L_1(\mu)$.

3. INTERPOLATION OF LINEAR OPERATORS BETWEEN $C$ AND $H^1$

Interpolation of operators between spaces of continuous functions and the space of Lipschitz functions was for the first time considered by Orlicz in 1954. More than ten years later analogous investigations were done by O’Neil [24], Peetre [32] and Semenov [36].

Let $\omega$ be a continuous function on $[0, \infty)$ with the following properties: $\omega$ is concave, increasing and $\omega(t) = 0$ iff $t = 0$.

By $H^\omega = H^\omega(I)$ we denote functions in $C(C = C(I)$ is the space of continuous on $I = [0, 1]$) for which the semi-norm

$$|x|_\omega = \sup_{s,t \in I, s \neq t} \frac{|x(s) - x(t)|}{\omega(|s - t|)}$$
is finite. The space $H^\omega$ is a Banach space with the norm

$$\|x\|_\omega = \|x\|_C + |x|_\omega.$$ 

If $\omega(t) = t^\alpha, 0 < \alpha \leq 1$ then we write $H^\alpha, |\cdot|_\alpha$ and $\|\cdot\|_\alpha$, in short.

**Theorem 3 (Orlicz).** The space $H^\omega(I)$ is an interpolation space between $C(I)$ and $H^1(I)$. More precisely, if $T : C(I) \to C(I)$ is any linear bounded operator with the norm $M_0$ which satisfies

$$(10) \quad |Tx|_1 \leq M_1 |x|_1 \quad \forall \in H^1(I),$$

then $T$ is bounded in $H^\omega(I)$ with

$$(11) \quad |Tx|_\omega \leq 3 \max(M_0, M_1) |x|_\omega \quad \forall \in H^\omega(I),$$

and so

$$(12) \quad \|Tx\|_\omega \leq 4 \max(M_0, M_1) \|x\|_\omega \quad \forall \in H^\omega(I).$$

If instead of (10) we have

$$(10') \quad \|Tx\|_1 \leq M_1 \|x\|_1 \quad \forall \in H^1(I),$$

then (12) holds with constant $3 + \max(\omega(1)^{-1}, 1)$ instead of 4.

**Proof.** First, let start with the classical approximation theorems of Jackson and Bernstein type. For $x \in C(I)$, the Steklov average functions $x_r \in H^1(I)(0 < r < 1)$ are defined by

$$x_r(t) = \begin{cases} 
  r^{-1} \int_t^{t+r} x(s) \, ds & \text{if } 0 \leq t \leq 1 - r, \\
  r^{-1} \int_{1-r}^1 x(s) \, ds & \text{if } 1 - r \leq t \leq 1.
\end{cases}$$

If $x \in H^\omega$, then (for $t', t'' \in I$ and $t' \neq t''$)

$$|x_r(t') - x_r(t'')|/|t' - t''|$$

$$\leq \sup_{t \in I} |x'_r(t)| = \sup_{t \in [0, 1-r]} r^{-1} |x(t + r) - x(t)|$$

$$\leq r^{-1} \omega(r) |x|_\omega,$$

and so

$$(13) \quad \|x_r\|_1 \leq r^{-1} \omega(r) |x|_\omega$$
Also,
\[ |x(t) - x_r(t)| = |x(t) - r^{-1} \int_{\min(t,1-r)}^{\min(t+r,1)} x(s)ds| \]
\[ \leq r^{-1} \int_{\min(t,1-r)}^{\min(t+r,1)} |x(t) - x(s)|ds \]
\[ \leq r^{-1} \int_{\min(t,1-r)}^{\min(t+r,1)} \omega(|t-s|)ds|x_\omega| \]
\[ \leq \omega(r)|x_\omega|, \]
thus
\[ \|x - x_r\|_C \leq \omega(r)|x_\omega|. \]  
(14)

On the other hand, if for some family \( \{x_r\}_{0 < r < 1} \) of functions in \( H^1(I) \) and some \( A, B > 0 \) we have
\[ |x_r|_1 \leq Ar^{-1}\omega(r) \quad \forall 0 < r < 1, \]
and
\[ \|x - x_r\|_C \leq B\omega(r) \quad \forall 0 < r < 1, \]
then \( x \in H^\omega(I) \) and \( |x_\omega| \leq A + 2B. \)

In fact, given \( 0 < h < 1 \) and \( n \) so that
\[ r^n \leq h < r^{n-1} \]
we get, for \( t \in [0, 1-h] \),
\[ |x(t+h) - x(t)| \leq |x(t+h) - x_{rn}(t+h)| + |x_{rn}(t+h) - x_{rn}(t)| + \]
\[ + |x(t) - x_{rn}(t)| \leq B\omega(r^n) + Ar^{-n}\omega(r^n)h + B\omega(r^n) \]
\[ \leq B\omega(h) + Ar^{-1}\omega(h) + B\omega(h) \leq (2B + Ar^{-1})\omega(h), \]
i.e., \( x \in H^\omega(I) \) and \( |x_\omega| \leq \inf_{0 < r < 1}(2B + Ar^{-1}) = A + 2B. \)

Now we are ready to prove the theorem. If \( x \in H^\omega(I) \), then putting \( y_r = T(x_r) \), where the \( x_r \) are the Steklov average functions, we have that \( y_r \in H^1(I) \) and
\[ |y_r|_1 = |T(x_r)|_1 \leq M_1|x_r|_1 \leq M_1r^{-1}\omega(r)|x_\omega|, \]
\[ \|T x - y_r\|_C = \|T(x - x_r)\|_C \leq M_0\|x - x_r\|_C \leq M_0\omega(r)|x_\omega|. \]
Therefore \( Tx \in H^\omega(I) \) with \( |Tx_\omega| \leq (2M_0 + M_1)|x_\omega| \), and from this we have (11) and (12).
If instead of (10) we have (10'), then
\[ |y_r|_1 \leq \|y_r\|_1 = \|T(x_r)\|_1 \leq M\|x_r\|_1 \]
\[ = M_1(\|x_r\|_C + |x_r|_1) \leq M_1(\|x\|_C + r^{-1}\omega(r)|x|_\omega) \]
\[ \leq M_1(r^{-1}\omega(r)/\omega(1)^{-1}\|x\|_C + r^{-1}\omega(r)/\omega(1)^{-1}M_1\|x\|_\omega) \]
\[ \leq r^{-1}\omega(r)\max(\omega(1)^{-1}, 1)M_1\|x\|_\omega \]
and so
\[ |Tx|_\omega \leq M_1 \max(\omega(1)^{-1}, 1)\|x\|_\omega + 2M_0|x|_\omega \]
\[ \leq [M_1 \max(\omega(1)^{-1}, 1) + 2M_0]\|x\|_\omega , \]
i.e.,
\[ \|Tx\|_\omega \leq [3M_0 + M_1 \max(\omega(1)^{-1}, 1)]\|x\|_\omega \]
\[ \leq [3 + \max(\omega(1)^{-1}, 1)]\max(M_0, M_1)\|x\|_\omega . \]

REMARKS 4. Orlicz [26] proved that (10') implies the inequality (12) with constant 7 + 2\omega(1)^{-1}, but his theorem was proved even in the vector-valued case, i.e., if X is a Banach space then \( H^\omega(I, X) \) is an interpolation space between \( C(I, X) \) and \( H^1(I, X) \). Our proof of Theorem 3 works without essential changes also in the vector-valued case.

5. O’Neil [24] considered situation for \( I = \mathbb{R} \) with the semi-norms in \( H^1 \) and \( H^\omega \). His family of functions \( \{x_r\}_{r > 0} \) was defined by
\[ x_r(t) = \int_{\mathbb{R}} x(t-s)k_r(s)ds, \quad k_r(s) = r^{-1}(1 - |s|/r)_+. \]

6. From the proof of Theorem 3 we also get that the Peetre \( K \)-functional for the pair \( \{C, H^1\} \) has a form
\[ K(t, x; C, H^1) \approx \omega(t, x) + \min(1, t)\|x\|_C , \]
where
\[ \omega(t, x) = \sup_{0 \leq h \leq t} \sup_{0 \leq s \leq 1-h} |x(s+h) - x(s)| , \]
and the interpolation spaces
\[ (C, H^1)_{\omega, \infty} := \{x \in C : \sup_{t > 0} K(t, x; C, H^1)/\omega(t) < \infty \} \]
are equal to \( H^\omega \) (see also Peetre [32]).

7. If for a given \( \omega_0 \) and \( \omega_1 \) we have
\[ \omega(t) = \omega_0(t)\omega_2(\omega_1(t)/\omega_0(t)) \quad \forall t > 0 \]

(15)
for some $\omega_2$, then using the reiteration theorem for the $K$-method with a function parameter we have

$$ (H^{\omega_0}, H^{\omega_1})_{\omega_2, \infty} = ((C, H^1)_{\omega_0, \infty}, (C, H^1)_{\omega_1, \infty})_{\omega_2, \infty} = (C, H^1)_{\omega, \infty} = H^\omega $$

Therefore, if $(15)$ holds then $H^\omega$ is an interpolation space between $H^{\omega_0}$ and $H^{\omega_1}$. In [16] even more general result is given.

**Problem 5.** Calculate the spaces of the complex method $[C, H^1]_\theta$ or a "more difficult" problem: describe all interpolation spaces between $C$ and $H^1$.

4. THE RIESZ-THORIN THEOREM FOR ORLICZ SPACES

Any interpolation theorem true for $L_p$-spaces is natural to generalized on Orlicz spaces. For example, it is natural to pose the following problem:

under what conditions is true that Orlicz space $L_\varphi$ is an interpolation space between Orlicz spaces $L_{\varphi_0}$ and $L_{\varphi_1}$?

The classical Riesz-Thorin interpolation theorem for $L_p$-spaces was extended to the case of Orlicz spaces by many authors (see [19] and the literature given there). The best generalization to Orlicz spaces was proved by Ovchinnikov [29], Krugljak (unpublished paper-cf. [19]), Maligranda [19] and Brudnyi-Krugljak [4].

**Theorem 4.** If $T : L_{\varphi_0}(\mu) + L_{\varphi_1}(\mu) \to L_{\psi_0}(\nu) + L_{\psi_1}(\nu)$ is a linear operator which is bounded from $L_{\varphi_0}(\mu)$ into $L_{\psi_0}(\nu)$ with the norm $M_0$ and bounded from $L_{\varphi_1}(\mu)$ into $L_{\psi_1}(\nu)$ with the norm $M_1$, then $T$ is bounded from $L_\varphi(\mu)$ into $L_\psi(\nu)$ with the norm $M \leq 26 \max(M_0, M_1)$, where

$$ \varphi^{-1} = \varphi_0^{-1} \rho(\varphi_1^{-1}/\varphi_0^{-1}), \quad \psi^{-1} = \psi_0^{-1} \rho(\psi_1^{-1}/\psi_0^{-1}) $$

and $\rho$ is any continuous function on $[0, \infty)$ which is concave, increasing and $\rho(t) = 0$ iff $t = 0$

Ovchinnikov proof was via his methods of interpolation and the proofs in [19] and [4] are via identification of the Calderón-Lozanovskii construction with the Gustavsson-Peetre interpolation method. Moreover, if we know additionally that $\lim_{t\to 0} \rho(t) = \lim_{t\to\infty} \rho(t) = 0$ then instead of 26 is possible to have 12 and if $\rho(t) = t^\theta, 0 < \theta < 1$, then the constant is 4 (cf. [19]).

**Problem 6.** Determine the exact constant $C$ in the interpolation inequality $M \leq C \max(M_0, M_1)$ of Theorem 4.

In Orlicz's fourth paper (joint with H. Hudzik and R. Urbański) there is a generalization of the Riesz-Thorin theorem with $\rho(t) = t^\theta, 0 \leq \theta \leq 1$, however for the vector-valued Musielak-Orlicz sequence spaces. Their proof uses the three lines theorem for subharmonic functions.
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Orlicz–Lorentz Spaces

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It is a great honor to be asked to write this article for the Proceedings of the Conference in honor of W. Orlicz. I cannot say that I knew him personally, but it is obvious from the many people I have met that knew him that he had a tremendous influence. Certainly, his ideas have found their way into much of my own research.

The purpose of this article is to summerize some recent results of the author about Orlicz-Lorentz spaces — function spaces that provide a common generalization of Orlicz spaces and Lorentz spaces.

Let us first introduce the background to these spaces. The most well known examples of Banach spaces are the \( L_p \) spaces. Their definition is very well known. We will restrict ourselves to function spaces on \([0, \infty)\) with Lebesgue measure \( \lambda \). If \( 1 \leq p \leq \infty \), then for any measurable function \( f \), the \( L_p \)-norm is defined to be

\[
\|f\|_p = \left( \int |f(x)|^p \, dx \right)^{1/p}
\]

for \( p < \infty \), and

\[
\|f\|_\infty = \text{ess sup}_{0 \leq x < \infty} |f(x)|
\]

for \( p = \infty \). The Banach space \( L_p \) is the vector space of all measurable functions \( f \) for which \( \|f\|_p \) is finite.

Now these spaces can be generalized in two different ways. The first generalization is due to Orlicz [O] (see also [Lu]). If \( F: [0, \infty) \to [0, \infty) \) is non-decreasing and convex with \( F(0) = 0 \), we define the Luxemburg norm of a measurable function \( f \) by

\[
\|f\|_F = \inf \left\{ c : \int F(|f(x)|/c) \, dx \leq 1 \right\}.
\]

We define the Orlicz space \( L_F \) to be those measurable functions \( f \) for which \( \|f\|_F \) is finite. We see that the Orlicz space \( L_F \) really is a true generalization of \( L_p \), at least for \( p < \infty \): if \( F(t) = t^p \), then \( L_F = L_p \) with equality of norms.

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COMPARISON OF ORLICZ–LORENTZ SPACES

In this article, we will not always require that the Luxemburg norm actually be a norm, that is, we will not always require the triangle inequality. For this reason, we will allow $F$ in the above definition to be a $\varphi$-function, namely, that $F$ be continuous and strictly increasing, and that

$$F(0) = 0, \quad \lim_{n \to \infty} F(t) = \infty.$$ 

However, we will often desire that the function $F$ has some control on its growth, both from above and below. For this reason we will often require that $F$ be dilatory, that is, for some $c_1, c_2 > 1$ we have $F(c_1 t) \geq c_2 F(t)$ for all $0 \leq t < \infty$, and that $F$ satisfy the $\Delta_2$-condition, that is, that $F^{-1}$ is dilatory.

The second collection of examples are the Lorentz spaces. These were introduced by Lorentz [Lo1], [Lo2]. If $f$ is a measurable function, we define the non-increasing rearrangement of $f$ to be

$$f^*(x) = \sup \{ t : \lambda(\{|f| \geq t\}) \geq x \}.$$ 

If $1 \leq q < \infty$, and if $w : (0, \infty) \to (0, \infty)$ is a non-increasing function, we define the Lorentz norm of a measurable function $f$ to be

$$\|f\|_{w,q} = \left( \int_0^\infty w(x) f^*(x)^q \, dx \right)^{1/q}.$$ 

Then the Lorentz space $\Lambda_{w,q}$ is defined to be the space of those measurable functions $f$ for which $\|f\|_{w,q}$ is finite. These spaces also represent a generalization of the $L_p$ spaces: if $w(x) = 1$ for all $0 \leq x < \infty$, then $\Lambda_{w,p} = L_p$ with equality of norms.

There is one, rather peculiar, choice of the function $w$ which turns out to be rather useful. If $1 \leq q \leq p < \infty$, we define the spaces $L_{p,q}$ to be $\Lambda_{w,q}$ with $w(x) = \frac{q}{p} x^{q/p-1}$. We can also allow $q > p$, but at the loss of the triangle inequality. A good reference for a description of these spaces is Hunt [H]. By a suitable change of variables, the $L_{p,q}$ norm may also be defined in the following fashion:

$$\|f\|_{p,q} = \left( \int_0^\infty |f^*(x^{p/q})|^q \, dx \right)^{1/q}.$$ 

Thus $L_{p,p} = L_p$ with equality of norms. The reason for this definition is that for any measurable set $A \in \mathcal{F}$, we have that $\|\chi_A\|_{p,q} = \|\chi_A\|_p = \lambda(A)^{1/p}$. Thus $L_{p,q}$ is a space identical to $L_p$ for characteristic functions, but ‘glued’ together in a $L_q$ fashion.

Now we come to the object of the article, the Orlicz–Lorentz spaces. These are a common generalization of the Orlicz spaces and the Lorentz spaces. They have been
studied by Mastyło (see part 4 of [My]), Maligranda [Ma], and Kamińska [Ka1], [Ka2], [Ka3]. For instance, Kamińska calculated many of the isometric properties for these spaces. However, this author’s work is concerned with isomorphic properties.

If $G$ is an Orlicz function, and if $w : [0, \infty) \to [0, \infty)$ is a non-increasing function, we define the Orlicz–Lorentz norm of a measurable function $f$ to be

$$
\|f\|_{w,G} = \inf \left\{ c : \int_0^\infty w(x)G\left(\frac{f^*(x)}{c}\right)dx \leq 1 \right\}.
$$

We define the Orlicz–Lorentz space $\Lambda_{w,G}$ to be the vector space of measurable functions $f$ for which $\|f\|_{w,G}$ is finite.

We shall not work with this definition of the Orlicz–Lorentz space, however, but with a different, equivalent definition that bears more resemblance to the spaces $L_{p,q}$. If $F$ and $G$ are $\varphi$-functions, we would like to define our spaces $L_{F,G}$ to satisfy the following properties:

i) that $\|\chi_A\|_{F,G} = \|\chi_A\|_F$ whenever $A$ is a measurable subset;

ii) that $L_{F,G}$ be glued together in a $L_G$ fashion.

It turns out that the required definition is the following. (In the sequel, $\tilde{F}(t)$ will always denote the function $1/F(1/t)$. Thus $\|\chi_A\|_F = \tilde{F}^{-1}(\lambda(A))$.)

**Definition:** If $F$ and $G$ are $\varphi$-functions, then we define the Orlicz–Lorentz functional of a measurable function $f$ by

$$
\|f\|_{F,G} = \left\| f^* \circ \tilde{F} \circ \tilde{G}^{-1} \right\|_G.
$$

We define the Orlicz–Lorentz space, $L_{F,G}$, to be the vector space of measurable functions $f$ for which $\|f\|_{F,G} < \infty$, modulo functions that are zero almost everywhere.

We also have the following definition corresponding to the $L_{p,\infty}$ spaces.

**Definition:** If $F$ is a $\varphi$-function, then we define the (weak-)Orlicz–Lorentz functional by

$$
\|f\|_{F,\infty} = \sup_{x \geq 0} \tilde{F}^{-1}(x)f^*(x).
$$

We define the Orlicz–Lorentz space, $L_{F,\infty}$, to be the vector space of measurable functions $f$ for which $\|f\|_{F,\infty} < \infty$, modulo functions that are zero almost everywhere.

We see that $L_{F,F} = L_F$ with equality of norms, and that if $F(t) = t^p$ and $G(t) = t^q$, then $L_{F,G} = L_{p,q}$, and $L_{F,\infty} = L_{p,\infty}$, also with equality of norms. For this reason, we shall also introduce the following notation: if $F(t) = t^p$, we shall write $L_{p,G}$ for $L_{F,G}$, and $L_{G,p}$ for $L_{G,F}$.
COMPARISON OF ORLICZ–LORENTZ SPACES

Now let us provide some examples. We define the modified logarithm and the modified exponential functions by

\[
\begin{align*}
\mathrm{lm}(t) &= \begin{cases} 
1 + \log t & \text{if } t \geq 1 \\
1/(1 + \log(1/t)) & \text{if } 0 < t < 1 \\
0 & \text{if } t = 0;
\end{cases} \\
\mathrm{em}(t) &= \mathrm{lm}^{-1}(t) = \begin{cases} 
\exp(t - 1) & \text{if } t \geq 1 \\
\exp(1 - (1/t)) & \text{if } 0 < t < 1 \\
0 & \text{if } t = 0.
\end{cases}
\end{align*}
\]

These functions are designed so that for large \( t \) they behave like the logarithm and the exponential functions, so that \( \mathrm{lm} 1 = 1 \) and \( \mathrm{em} 1 = 1 \), and so that \( \overline{\mathrm{lm}} = \mathrm{lm} \) and \( \overline{\mathrm{em}} = \mathrm{em} \). Then the functions \( t^p(\mathrm{lm} t)^\alpha \) and \( \mathrm{em}(t^p) \) are \( \varphi \)-functions whenever \( 0 < p < \infty \) and \( -\infty < \alpha < \infty \). If the measure space is a probability space, then the Orlicz spaces created using these functions are also known as Zygmund spaces, and the Orlicz–Lorentz spaces \( L_{t^p(\mathrm{lm} t)^\alpha},q \) and \( L_{\mathrm{em}(t^p),q} \) are known as Lorentz–Zygmund spaces (see, for example, [B–S]).

**Comparison Results**

A large part of my research on these spaces has asked the question: what are necessary and sufficient conditions on \( F_1, F_2, G_1 \) and \( G_2 \) so that the spaces \( L_{F_1,G_1} \) and \( L_{F_2,G_2} \) are equivalent, that is, that there is some constant \( c < \infty \) such that

\[
c^{-1} \|f\|_{F_1,G_1} \leq \|f\|_{F_2,G_2} \leq c \|f\|_{F_1,G_1}.
\]

In answering these questions, it is necessary to assume that \( G_1 \) and \( G_2 \) be dilatory, and satisfy the \( \Delta_2 \)-condition. In all our general discussions, we shall take this as given.

First, by considering characteristic functions, it is easy to see that it must be that \( F_1 \) and \( F_2 \) are equivalent as \( \varphi \)-functions, that is, there is a constant \( c < \infty \) such that \( F_1(c^{-1} t) \leq F_2(t) \leq F_1(ct) \) for all \( 0 \leq t < \infty \). In this manner, it is easy to see that without loss of generality, we may take \( F_1 = F_2 \). In fact, it is not hard to show that we are really asking about the equivalence of \( L_1 \) and \( L_{1,H} \), where \( H = G_1 \circ G_2^{-1} \) or \( H = G_2 \circ G_1^{-1} \).

Results along these lines have already been obtained by G. Lorentz, and also by Y. Raynaud. I will take the liberty of translating their results into my notation. (In so doing, it may not be entirely obvious that their result as they state it, and as it is stated here, are actually the same.)

To state these results we will require some more notation. We will say that a \( \varphi \)-function \( F \) is an \( N \)-function if it is equivalent to a \( \varphi \)-function \( F_0 \) such that \( F_0(t)/t \) is strictly increasing, \( F_0(t)/t \to \infty \) as \( t \to \infty \), and \( F_0(t)/t \to 0 \) as \( t \to 0 \). We will say that a \( \varphi \)-function \( F \) is complementary to a \( \varphi \)-function \( G \) if for some \( c < \infty \) we have

\[
c^{-1} t \leq F^{-1}(t) \cdot G^{-1}(t) \leq ct \quad (0 \leq t < \infty).
\]
If $F$ is an $N$-function, we will let $F^*$ denote the (unique up to equivalence) function complementary to $F$.

Our definition of a complementary function differs from the usual definition. If $F$ is an $N$-function that is convex, then the complementary function is usually defined by $F^*(t) = \sup_{s \geq 0} (st - F(s))$. However, it is known that $t \leq F^{-1}(t) \cdot F^{-1}(t) \leq 2t$ (see [K–R]). Thus our definition is equivalent.

Finally, we will say that an $N$-function $H$ satisfies condition $(J)$ if

$$\|1/\tilde{H}^{-1}\|_{H^*} < \infty.$$ 

(I call it condition $(J)$ for personal reasons.)

Now we are ready to give the result of G. Lorentz [Lo3].

**Theorem 1.** Suppose that $H$ is an $N$-function. Then the following are equivalent.

1) $L_1$ and $L_{1,H}$ are equivalent.

2) $H$ satisfies condition $(J)$.

What kinds of $N$-functions satisfy condition $(J)$? They are functions that satisfy growth conditions that make it ‘close’ to the identity function. The reader might like to verify that $t(\log t)^\alpha$ satisfies this condition when $\alpha > 0$. Lorentz gave the following example:

$$H(t) = \begin{cases} t^{1+\frac{1}{1+\log(1+\log t)}} & \text{if } t \geq 1 \\
1^{-\frac{1}{1+\log(1-\log t)}} & \text{if } t \leq 1. \end{cases}$$

In fact, we will give another characterization that shows that this example is, in some sense, on the “boundary” of satisfying condition $(J)$.

Raynaud’s result [R] allows one to drop the assumption that $H$ is an $N$-function, but at the cost of making the implication go only one way.

**Theorem 2.** Suppose that $H$ is a $\varphi$-function. Suppose that there exist $N$-functions $K$ and $L$ satisfying condition $(J)$ such that $H = K \circ L^{-1}$ (or $H = K^{-1} \circ L$). Then $L_1$ and $L_{1,H}$ are equivalent.

As applications, one may show that if $0 < p < \infty$ and $-\infty < \alpha < \infty$, then $L_{1p}(\log t)^\alpha$ and $L_{1p}(\log t)^\alpha, p$ are equivalent, and that if $\beta > 0$, then $L_{em(\beta)}$ and $L_{em(\beta),\infty}$ are equivalent. These were shown for probability spaces by Bennett and Rudnick [B–R] (see also [B–S]).

The author’s contribution was to show that the converse result to Theorem 2 holds.

**Theorem 3.** Suppose that $H$ is a $\varphi$-function such that $L_1$ and $L_{1,H}$ are equivalent. Then the following are true.

1) There exist $N$-functions $K$ and $L$ satisfying condition $(J)$ such that $H = K \circ L^{-1}$.

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ii) There exist \( N \)-functions \( K \) and \( L \) satisfying condition \((J)\) such that \( H = K^{-1} \circ L \).

In fact there are many more equivalent conditions, and we will give some more later. We will not prove any results here — the interested reader should consult [Mo1]. However, we will explain some of the ideas behind them.

First we will describe the simple comparison principles for Orlicz–Lorentz spaces. If the reader has studied Lorentz spaces, he will already know that \( \|f\|_{p,q_1} \leq \|f\|_{p,q_2} \) whenever \( q_1 \geq q_2 \) (see [H]). In our more general setting, we have the following result: if \( F \) is equivalent to a convex function, then \( \|f\|_F \leq c \|f\|_{F,1} \) for all measurable \( f \). In fact, it is quite easy to show that if \( \| \cdot \| \) is any norm (the triangle inequality is essential here) such that

\[
|f| \leq \chi A \Rightarrow \|f\| \leq c_1 \tilde{F}^{-1}(\lambda(A)),
\]

then \( \|f\| \leq c_2 \|f\|_{F,1} \).

From this, we can deduce the following result. Let us say that \( G_1 \) is equivalently less convex than \( G_2 \) (in symbols \( G_1 \prec G_2 \)) if \( G_2 \circ G_1^{-1} \) is equivalent to a convex function. Then

\[
G_1 \prec G_2 \Rightarrow \|f\|_{1,G_1} \geq c^{-1} \|f\|_{1,G_2}.
\]

However, we can see from Theorems 1 and 2 that this is not the whole story. If we desire a converse to this implication, we will have to soften the notion of ‘less convex than’ to ‘almost less convex than.’ It turns out that we can precisely characterize this notion of ‘almost convexity.’

Before doing this, let us discuss what it means for a \( \varphi \)-function to be equivalent to a convex function. Suppose we are given a fixed number \( a > 1 \). It is quite easy to see that a \( \varphi \)-function \( G \) is completely determined, up to equivalence, by the values \( G(a^n) \) for \( n \in \mathbb{Z} \). In this way, it can be easily shown that a \( \varphi \)-function \( G \) is equivalent to a convex function if and only there exists numbers \( a > 1 \) and \( N \in \mathbb{N} \) such that for all \( m \in \mathbb{N} \), we have that

\[
G(a^{n+m}) \geq a^{m-N} G(a^n)
\]

for all \( n \in \mathbb{Z} \). (Here \( \mathbb{N} = \{1, 2, 3, \ldots \} \).)

It turns out that the correct definition for ‘almost convex’ is the following.

**Definition:** Let \( G \) be a \( \varphi \)-function. We say that \( G \) is almost convex if there are numbers \( a > 1, b > 1 \) and \( N \in \mathbb{N} \) such that for all \( m \in \mathbb{N} \), the cardinality of the set of \( n \in \mathbb{Z} \) such that we do not have \( G(a^{n+m}) \geq a^{m-N} G(a^n) \) is less than \( b^m \).

In the same way, one can get notions of almost concave, almost linear, etc. It turns out that an \( N \)-function \( H \) satisfies condition \((J)\) if and only if \( H^{-1} \) is almost convex.
Using these ideas, it is then possible to prove Theorem 3, and indeed to get the following result, that gives the desired necessary and sufficient conditions for Orlicz–Lorentz spaces to be equivalent.

**Theorem 4.** Suppose that \( F_1, F_2, G_1 \) and \( G_2 \) are \( \varphi \)-functions such that at least one of \( G_1 \) and \( G_2 \) are dilatory, and at least one of \( G_1 \) and \( G_2 \) satisfy the \( \Delta_2 \)-condition. Then the following are equivalent statements.

i) \( L_{F_1, G_1} \) and \( L_{F_2, G_2} \) are equivalent.

ii) \( F_1 \) and \( F_2 \) are equivalent, and there exist \( N \)-functions \( H \) and \( K \) that satisfy condition (J) such that \( G_1 \circ G_2^{-1} = H \circ K^{-1} \).

iii) \( F_1 \) and \( F_2 \) are equivalent, and there exist \( N \)-functions \( H \) and \( K \) that satisfy condition (J) such that \( G_1 \circ G_2^{-1} = K^{-1} \circ H \).

iv) \( F_1 \) and \( F_2 \) are equivalent, and \( G_1 \circ G_2^{-1} \) is almost convex, and \( G_2 \circ G_1^{-1} \) is almost convex.

**Is Every R.I. Space Equivalent to an Orlicz–Lorentz Space?**

Or more precisely, does there exist a rearrangement invariant space \( X \) such that the \( \| \cdot \|_X \) is not equivalent to any Orlicz–Lorentz norm on the space of simple functions? It turns out that we can find an example to show that this can happen. To do this, we use the following result, which is a corollary of the proof of Theorem 4. As before, we refer the reader to [Mo1] for details.

**Theorem 5.** Let \( F_1, F_2, G_1 \) and \( G_2 \) be \( \varphi \)-functions. Suppose that one of \( G_1 \) or \( G_2 \) is dilatory, and that one of \( G_1 \) or \( G_2 \) satisfies the \( \Delta_2 \)-condition. Then the following are equivalent.

i) \( L_{F_1, G_1} \) and \( L_{F_2, G_2} \) are equivalent.

ii) For some \( c < \infty \) we have that \( c^{-1} \| f \|_{F_1, G_1} \leq \| f \|_{F_2, G_2} \leq c \| f \|_{F_1, G_1} \) whenever \( f \) is of the following form: there exist \( 0 = a_0 < a_1 < a_2 < \ldots < a_n \) such that

\[
F \circ f^*(x) = \begin{cases} 
1/a_i & \text{if } a_{i-1} \leq x < a_i \text{ and } 1 \leq i \leq n \\
0 & \text{otherwise}.
\end{cases}
\]

Thus to compare two Orlicz–Lorentz spaces, we need only compare their norms on a certain class of test functions. Now it is easy to prove the desired result.

**Theorem 6.** There is a rearrangement invariant Banach space \( X \) such that for every Orlicz–Lorentz space \( L_{F,G} \), the norms \( \| \cdot \|_X \) and \( \| \cdot \|_{F,G} \) are inequivalent on the vector space of simple functions.

**Proof:** We define the following norm for measurable functions \( f \):

\[
\| f \|_X = \sup \| fg \|_1 / \| g \|_2,
\]

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where the supremum is over all \( g \) of the following form: there exist \( 0 = a_0 < a_1 < a_2 < \ldots < a_n \) such that

\[
g^*(x) = \begin{cases} \frac{1}{\sqrt{a_i}} & \text{if } a_{i-1} < x < a_i \text{ and } 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}
\]

Then it is easy to see from Theorem 5 that if \( X \) is equivalent to an Orlicz–Lorentz space, then it must be equivalent to \( L_2 \). That this is not the case is easily shown by the following example:

\[
f(x) = \frac{1}{\sqrt{x \log x}} \quad \text{for } x \geq 2.
\]

Then \( \|f\|_X < \infty \), whereas \( \|f\|_2 = \infty \). \( \square \)

**Boyd Indices of Orlicz–Lorentz Spaces**

In studying a particular rearrangement invariant space, it is very important to know its Boyd indices. Even very obvious questions, like whether it is equivalent to a normed space, or whether it is \( p \)-convex/\( q \)-concave, cannot be answered except with a knowledge of these indices.

As their name suggests, they were first studied by Boyd [Bo]. We will take our definition from [L–T]. Their definition differs from that usually used: most references reverse the words ‘upper’ and ‘lower’, and use the reciprocals of the indices used here.

Essentially, they describe the norms of the following operators: for each \( a > 0 \) we let \( d_a f(x) = f(ax) \). The lower Boyd index is defined to be

\[
p(X) = \sup \left\{ p : \text{for some } c < \infty \text{ we have } \|d_a\|_{X \to X} \leq ca^{-1/p} \text{ for } a < 1 \right\},
\]

and the upper Boyd index is

\[
q(X) = \inf \left\{ q : \text{for some } c < \infty \text{ we have } \|d_a\|_{X \to X} \leq ca^{-1/q} \text{ for } a > 1 \right\}.
\]

The reader should appreciate that \( p(L_{p,q}) = q(L_{p,q}) = p \).

The hope is that it should be possible to calculate the Boyd indices of \( L_{F,G} \) simply from knowledge of some appropriate index of \( F \). In fact, this was the question posed by Maligranda [Ma]. What are the appropriate indices? For a \( \varphi \)-function \( F \), we define the lower Matuszewska–Orlicz index to be

\[
p_m(F) = \sup \left\{ p : \text{for some } c > 0 \text{ we have } F(at) \geq ca^p F(t) \text{ for } 0 \leq t < \infty \text{ and } a > 1 \right\},
\]

and the upper Matuszewska–Orlicz index to be

\[
q_m(F) = \inf \left\{ q : \text{for some } c < \infty \text{ we have } F(at) \leq ca^q F(t) \text{ for } 0 \leq t < \infty \text{ and } a > 1 \right\}.
\]
Thus, for example, $p_m(T^p) = q_m(T^p) = p$. Maligranda's conjecture is the following: is $p(L_{F,G}) = p_m(F)$ and $q(L_{F,G}) = q_m(F)$?

Without going into details, I was able to show that this is not the case. Briefly, the example is $L_{1,G}$, where $G$ is a $\varphi$-function that spends some of the time behaving like $T^p$, and some of the time behaving like $T^q$. We refer the reader to [Mo2] for more details. (The first example of a rearrangement invariant space where this sort of thing happened is due to Shimogaki [Sh].)

However, it is possible to obtain the following result without undue stress.

**Proposition 7.** Let $F$ and $G$ be $\varphi$-functions. Then

i) $p_m(F) \geq p(L_{F,G}) \geq p_m(F \circ G^{-1}) p_m(G) \geq p_m(F) p_m(G)/q_m(G)$;

ii) $q_m(F) \leq q(L_{F,G}) \leq q_m(F \circ G^{-1}) q_m(G) \leq q_m(F) q_m(G)/p_m(G)$.

We are then left with the following question. Given $F$ and $G$, how exactly is one to calculate the Boyd indices of $L_{F,G}$? The author does have some idea for how to approach this problem, at least for giving necessary and sufficient conditions for the indices of $L_{1,G}$ to be 1. The idea is simple: we see that if $0 < a < \infty$, then $a \|d_a f\|_{1,G} = \|f\|_{1,G_a}$, where $G_a(t) = G(at)$. Then the problem of determining the Boyd indices becomes a problem of comparing two Orlicz–Lorentz spaces, and the methods from the above section should apply. One day, the author will get around to checking these ideas out. But if anyone else would like to do this, they can, and the author won't mind. Then they will have the problem of finding a journal that will accept results from this tiny corner of mathematics.

Finally, I would like to mention some very recent work of Bastero and Ruiz [Ba–R]. They prove some results about the Hardy transform on Orlicz–Lorentz spaces. If one looks hard enough at what they did, and then twists the way they state the results, one can obtain fairly sharp estimates for Boyd indices in the following manner. Given $\varphi$-functions $F$ and $G$, we define the *modular lower and upper Boyd indices* of $L_{F,G}$ as follows:

$$p_{\text{mod}}(L_{F,G}) = \sup \left\{ p : \text{for some } c < \infty \text{ we have} \right\}$$

$$\int G(f^*(aF \circ G^{-1}(x))) \, dx \leq \int G(c a^{-1/p} f^*(\tilde{F} \circ \tilde{G}^{-1}(x))) \text{ for } a < 1 \right\},$$

$$q_{\text{mod}}(L_{F,G}) = \inf \left\{ q : \text{for some } c < \infty \text{ we have} \right\}$$

$$\int G(f^*(aF \circ G^{-1}(x))) \, dx \leq \int G(c a^{-1/q} f^*(\tilde{F} \circ \tilde{G}^{-1}(x))) \text{ for } a > 1 \right\}.$$

Then we have the following result.

**Theorem 8.** Let $F$ and $G$ be $\varphi$-functions. Then
COMPARISON OF ORLICZ–LORENTZ SPACES

i) $p_{\text{mod}}(L_{F,G}) = p_m(F \circ G^{-1})p_m(G)$;

ii) $q_{\text{mod}}(L_{F,G}) = q_m(F \circ G^{-1})q_m(G)$.

THE DEFINITION OF TORCHINSKY AND RAYNAUD

Finally, we mention that there is another possible definition for Orlicz–Lorentz spaces, first given by Torchinsky [T], and investigated in detail by Raynaud [R]. We define

$$
\|f\|^T_{F,G} = \left\| \hat{F}^{-1}(e^x)f^*(e^x) \right\|_G,
$$

and call the corresponding space $L^T_{F,G}$ (my notation). Raynaud showed that if $F$ is dilatory and satisfies the $\Delta_2$-condition, and if $G$ is dilatory, then

$$
\|\chi_A\|^T_{F,G} \approx \hat{F}^{-1}(\lambda(A)).
$$

Thus these spaces are really quite a good contender for a possible alternative definition. Also, the problems that I considered are very easy to solve for these spaces. Raynaud showed that if $F_1$ and $F_2$ are dilatory and satisfy the $\Delta_2$-condition, and if $G_1$ and $G_2$ are dilatory, then $L^T_{F_1,G_1}$ and $L^T_{F_2,G_2}$ are equivalent if $F_1$ and $F_2$ are equivalent, and the sequence spaces $l_{G_1}$ and $l_{G_2}$ are equivalent. The converse result is also easy to show.

Also, the Boyd indices of these spaces are much easier to compute. If $F$ is dilatory and satisfies the $\Delta_2$-condition, and if $G$ is dilatory, then $p(L_{F,G}) = p_m(F)$, and $q(L_{F,G}) = q_m(G)$.

The only problem with these spaces is that we do not always have that $L^T_{F,F}$ is equivalent to the Orlicz space $L_F$. 

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THE ORLICZ-PETTIS THEOREM FAILS FOR VECTOR VALUED $H^p$-SPACES
by
M. Nawrocki

We recall that a topological vector space $X=(X,\tau)$ whose
topological dual $X'$ separates the points is said to have the
Orlicz-Pettis Property (OPP), if each weakly subseries convergent
series in $X$ (i.e., a series $\sum x_n$ in $X$ such that $\text{weak-lim}_{n \to \infty} \sum_{j=1}^n x_j$ exists for each increasing sequence \{k_j\} of positive
integers) is $\tau$-convergent (i.e., the Orlicz-Pettis Theorem holds
in $X$). It is well known that all locally convex spaces and all
separable F-spaces (complete metrizable t.v.s.) with separating
duals have the OPP (we refer to [2] for information on the
Orlicz-Pettis theorem). For a long time it was left open if the
Orlicz-Pettis Theorem can be extended to the class of
nonseparable F-spaces with separating duals. Recently, the
author showed that the Orlicz-Pettis Theorem fails for weak-$\ell^p$
sequence spaces $\ell(p,\infty)$ if $0 < p \leq 1$ (cf. [6]), Lumer-Hardy spaces
$LH^n(B_n)$ of the unit ball $B_n$ in $\mathbb{C}^n$ if $0 < p < 1$ and $n > 1$ [7], and
Lumer-Hardy algebras of the balls $B_n$ and the unit polydiscs $U^n$ in
$\mathbb{C}^n$ for $n > 1$ [9]. In the sequel we present a proof that if $0 < p
< 1$ then $H^p$-spaces of functions with values in complex Banach
spaces containing copies of $c_0$ have not the OPP (cf. [8]).

Throughout this paper we assume that $X = (X,\| \cdot \|)$ is a
complex Banach space. Given $0 < p < \infty$ we shall denote by $H^p(X)$
the space of $X$-valued holomorphic functions $f$ on the unit disc $U
= \{ z \in \mathbb{C} : |z| < 1 \}$ such that
$$\| f \|_{H^p(X)} = \sup_{0 < r < 1} \left( \frac{1}{\mathcal{T}} \int \| f(r\omega) \|_p \, dm(\omega) \right)^{1/p} < \infty,$$
where $m$ is the normalized Lebesgue measure on the unit circle $\mathcal{T} = \partial U$.

If $p \geq 1$ then $H^p(X)$ is a Banach space while for $0 < p < 1$ it
is a $p$-Banach space whose topological dual separates the points.
We recall that a $p$-norm on a vector space $E$ is a functional
$\| \cdot \| : E \rightarrow [0,\infty)$ satisfying
\[ \| \alpha x \| = |\alpha| \| x \| \]
\[ \| x + y \| \leq \| x \| + \| y \| \]
for each scalar \( \alpha \) and \( x, y \in E \). (\( E, \| \cdot \| \)) is called a \( p \)-Banach space if the metric space \((E,d)\), where \( d(x,y) = \| x - y \|_p \), is complete. On each \( p \)-Banach space \((E,\| \cdot \|)\) the Minkowski functional \( \| \cdot \| \) of the convex hull of the unit ball \( B = \{ x \in E : \| x \| \leq 1 \} \) is a semi-norm (a norm if \( E' \) separates the points) which is called the canonical semi-norm (norm) on \( E \). It is easily seen that the spaces \((E,\| \cdot \|)\), \((E,\| \cdot \|^\wedge)\) have the same dual space. Moreover, \((E,\| \cdot \|)\) is locally convex, so it has the OPP. Thus, in order to prove that \((E,\| \cdot \|)\) has the OPP it suffices to show that each \( \| \cdot \|^\wedge \)-subseries convergent series in \( E \) is \( \| \cdot \|\)-convergent.

The reader is referred to [3] for wide information on \( p \)-Banach spaces.

**Theorem:** If a Banach space \( X \) contains a copy of \( c_0 \), then \( H^p(X), 0 < p < 1, \) does not have the Orlicz-Pettis Property.

**Proof:** Let \((x_n)\) be a sequence in \( X \) equivalent to the unit vector basis in \( c_0 \). For each \( n \in \mathbb{N} \) define \( f_n(z) = x_n z^n, \ z \in \mathbb{U} \). Obviously, the series \( \sum f_n \) is not convergent in \( H^p(X) \). Let \( \| \cdot \|^\wedge \in H^p(X) \) be the canonical norm on \( H^p(X) \). We shall prove that

\[ \sum f_n \] is \( \| \cdot \|^\wedge \)-subseries convergent.

Let \( \sum f_n \) be an arbitrary subseries of \( \sum f_n \). Then

\[ g_m(z) = \sum_{k=m}^{\infty} x_n z^k \]
is an \( X \)-valued analytic function on \( \mathbb{U} \), for each \( m \in \mathbb{N} \). Since \((x_n)\) is equivalent to the unit vector basis in \( c_0 \), so \( \sup\{|g_m(z)|: z \in \mathbb{U}, m \in \mathbb{N}\} = C < \infty \).

For each \( m \in \mathbb{N} \) define an operator \( T_m : H^p \to H^p(X) \) by \( T_m h = h \cdot g_m \). It is easily seen that

\[ \| T_m h \|^\wedge \leq \| T_m h \|_p \leq C \| h \|_p \]
so

\[ \| T_m h \|^\wedge \leq C \| h \|^\wedge \]
for each \( h \in H^p \). However,

\[ T_m z^m = \sum_{k=m}^{\infty} f_k \]
so in order to prove that the subseries $\sum f_{n_k}$ is $\parallel \cdot \parallel_{H^p(X)}$-convergent it is enough to show that $\parallel z^m \parallel_{H^p} \to 0$ if $m \to \omega$. But if it is known that the canonical norm $\parallel \cdot \parallel_{H^p}$ on $H^p$ is equivalent to the norm $\parallel \cdot \parallel_{H^p(X)}$ defined by

$$\parallel f \parallel_{H^p} = \int |f(z)| (1 - |z|)^{1/p-2} dA(z)$$

where $A$ is the normalized Lebesgue measure on $U$, and $f \in H^p$ (cf. [1]). Now, $\parallel z^m \parallel \to 0$ by the dominate convergence theorem.

**Proposition:** If $H^p(X)$ does not have the Orlicz-Pettis Property then it contains a copy of $l_\infty$.

**Proof:** Suppose that $H^p(X)$ does not have the Orlicz-Pettis property and let $\sum f_n$ be a series in $H^p(X)$ which is weakly subseries convergent but nonconvergent in the original topology of $H^p(X)$. Then $\sum f_n$ is $\parallel \cdot \parallel_{H^p(X)}$-subseries convergent. The topology $\mu$ defined in $H^p(X)$ by $\parallel \cdot \parallel_{H^p(X)}$ is stronger than the compact-open topology $\kappa$, so $\sum f_n$ is $\kappa$-subseries convergent. Therefore, the formula $\nu(A) = \sum_{n \in A} f_n$ defines a $H^p(X)$-valued measure on the family of all subsets of $\mathbb{N}$, which is $\kappa$-countably additive (and so it is $\kappa$-bounded). The closed balls in $H^p(X)$ are $\kappa$-closed, so by [4] Theorem 3, the range of $\nu$ is bounded in $H^p(X)$. Since the series $\sum f_n$ is not convergent in $H^p(X)$, so the measure $\nu$ is not exhaustive if $H^p(X)$ is equipped with its own topology. Finally, by [5] Corollary A p. 242, $H^p(X)$ contains a copy of $l_\infty$.

**Conjecture:** $H^p(X)$ contains a copy of $l_\infty$ if and only if $X$ contains a copy of $c_0$.

**References**


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GENERALIZED CONVEXITY AND THE MAZUR-ORLICZ THEOREM

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Introduction. The classical consistency theorem due to Mazur and Orlicz [20] from 1953 has become an important tool in different branches of analysis and its applications. In our talk at the Orlicz Memorial Conference at the University of Mississippi in March 1991, we took the opportunity to discuss various aspects of what is now called the Mazur-Orlicz theorem. The present paper is an extended version of this talk; it includes additional references, further applications as well as complete proofs of some new results.

The section headlines indicate the program of this paper. First we discuss briefly the pioneering work of Mazur and Orlicz in connection with interpolation theorems. We also indicate how their result developed over the years in various directions. Then we focus on the main Mazur-Orlicz type theorem of our recent paper [24] and sketch some of the immediate applications related to minimax theory.

We believe that the Mazur-Orlicz theorem has been fundamental for the systematic development of a powerful theory of sublinear functionals. For the sake of illustration, we present some basic results of this theory in section 2. Here, the emphasis is on the decomposition of linear functionals following König [13]. This approach leads naturally to our general version of the Strassen disintegration theorem from [22]. The purpose of section 3 is to collect some typical, but perhaps not well-known sublinear functionals from different areas. Our examples are taken from measure theory, Hardy algebras, and network theory.

In the final section, we present a new version of the Mazur-Orlicz theorem for a certain class of generalized convex operators. We also describe some situations where this new sandwich type theorem turns out to be useful. In particular, the theory presented here subsumes and generalizes earlier results due to Fuchssteiner and König [5] and also recent work of Martellotti and Salvadori [19].

The list of publications at the end of this paper contains only papers immediately connected with the topics discussed here. Our presentation is very much in the spirit of the monographs [6] and [16], where the interested reader will find further information and references. The present paper is also closely related to our talk at the Conference on Optimization and Convex Analysis at the University of Mississippi in March 1989, but we tried to minimize the overlap with our paper [23].
1. From Sandwich to Minimax Theorems. Basically, the Mazur-Orlicz theorem in question is about the existence of linear functionals on a real vector space, which are dominated by a given sublinear functional and, at the same time, satisfy certain inequality constraints. Thus the Mazur-Orlicz theorem is closely related to and actually an extension of the Hahn-Banach theorem. The classical version is somewhat hidden as result 2.41 in the second long paper by Mazur and Orlicz on metric linear spaces [20]. The following theorem is an equivalent version of their result and better suited for the generalizations we have in mind. Recall that a real-valued functional \( \vartheta \) on a real vector space \( E \) is said to be sublinear if \( \vartheta(u + v) \leq \vartheta(u) + \vartheta(v) \) and \( \vartheta(tu) = t\vartheta(v) \) holds for all \( u, v \in E \) and all real \( t \geq 0 \). Thus the sublinear functionals are precisely the Banach functionals in the terminology of Mazur and Orlicz [20].

1.1. Mazur-Orlicz Theorem. Let \( \vartheta : E \to \mathbb{R} \) be a sublinear functional on a real vector space \( E \), and consider an arbitrary mapping \( \rho : K \to \mathbb{R} \) on a non-empty subset \( K \) of \( E \). Then the following assertions are equivalent:

(a) There exists a linear functional \( \varphi : E \to \mathbb{R} \) with \( \rho \leq \varphi \) on \( K \) and \( \varphi \leq \vartheta \) on \( E \).

(b) \[ \sum_{k=1}^{r} t_k \rho(u_k) \leq \vartheta(\sum_{k=1}^{r} t_k u_k) \] for all \( u_1, \ldots, u_r \in K \) and all real \( t_1, \ldots, t_r \geq 0 \).

The original proof of this result in [20] was elementary, but lengthy and not very transparent. Shorter proofs were given by Sikorski [31], Pták [29], Simons [32], and various other authors in similar or somewhat more general situations. For instance, there are useful extensions and variations of the Mazur-Orlicz theorem for additive functionals on abelian semigroups. These results date back, in chronological order, to Aumann [1], Kaufman [9], Kranz [17], and Fuchssteiner [4]. Further information on these aspects and a variety of applications can be found in [6] and also in an interesting paper by Kindler [10]. For certain vector-valued versions of the Mazur-Orlicz theorem, we refer again to the monograph [6] and also to Peressini [28] and Zowe [35]. Moreover, let us mention the abstract version of the Hahn-Banach theorem developed by Rodé [30]. A simplified proof of this powerful result has recently been provided by König [15]. In the following, we shall restrict ourselves to the more conventional vector space situation. For completeness, we include a short proof of the classical Mazur-Orlicz theorem.

Proof of 1.1. Obviously (a) implies (b). To show the converse, we follow a beautiful idea introduced by Pták [29] and consider the functional \( \psi \) given by

\[
\psi(x) := \inf \left\{ \vartheta(x + \sum_{k=1}^{r} t_k u_k) - \sum_{k=1}^{r} t_k \rho(u_k) : u_1, \ldots, u_k \in K \text{ and } t_1, \ldots, t_r \geq 0 \right\}
\]

for all \( x \in E \). Then it is easily deduced from (b) that \( \psi \) is real-valued and sublinear on \( E \). Moreover, the linear functionals dominated by the auxiliary functional \( \psi \) are easily seen to be precisely the linear functionals satisfying condition (a). Since every sublinear functional is known to dominate at least one linear functional, the assertion follows.

Let us note in passing that this approach immediately carries over to the case of operators with values in a Dedekind complete ordered vector space; see for instance [28].
Another equivalent version of the Mazur-Orlicz theorem is the following sandwich type theorem: if a sublinear functional \( \vartheta \) on a real vector space \( E \) dominates a concave functional \( \rho \) on a convex subset \( K \) of \( E \), then there exists a linear functional \( \varphi \) on \( E \) such that \( \rho \leq \varphi \) on \( K \) as well as \( \varphi \leq \vartheta \) on \( E \). Starting with König [11], considerable efforts have been made to weaken the convexity assumption in this context [5], [14], [16]. The most general result in this direction has recently been published in [24]:

1.2. Sandwich Theorem. Let \( \vartheta : E \to F \) be a sublinear operator from a real vector space \( E \) into a Dedekind complete vector lattice \( F \), and consider an arbitrary mapping \( \rho : K \to F \) on a non-empty subset \( K \) of \( E \) such that \( \rho \leq \vartheta \) on \( K \). Moreover, assume that for some pair of real numbers \( \alpha, \beta > 0 \) and some \( u \in F_+ \) the following condition is fulfilled:

\[
(1) \quad \text{For all } x, y \in K \text{ and all } \varepsilon > 0 \text{ there exists some } z \in K \text{ such that} \]
\[
\vartheta(z - \alpha x - \beta y) \leq \rho(z) - \alpha \rho(x) - \beta \rho(y) + \varepsilon u.
\]

Then there exists a linear operator \( \varphi : E \to F \) such that \( \rho \leq \varphi \) on \( K \) and \( \varphi \leq \vartheta \) on \( E \).

Of course, condition (1) may be considered as a weak convexity assumption: if the set \( K \) is convex and the function \( \rho : K \to F \) is concave, then this condition is obviously fulfilled with an arbitrary choice of real numbers \( \alpha, \beta > 0 \) satisfying \( \alpha + \beta = 1 \) and even with the trivial choice \( u = 0 \). Moreover, in the classical case \( F = \mathbb{R} \), condition (1) is equivalent to the following König type convexity condition [5], [14]: there exists a pair of real numbers \( \alpha, \beta > 0 \) such that

\[
(2) \quad \inf \{ \vartheta(z - \alpha x - \beta y) - \rho(z) + \alpha \rho(x) + \beta \rho(y) : z \in K \} \leq 0 \text{ for all } x, y \in K.
\]

Actually, even in the vector-valued setting condition (1) always implies (2), since it is well-known and easily seen that the order of a Dedekind complete vector lattice is Archimedean [8]. On the other hand, condition (1) is certainly fulfilled, if the following more attractive condition holds for some pair of real numbers \( \alpha, \beta > 0 \):

\[
(3) \quad \text{For all } x, y \in K \text{ there is a } z \in K \text{ such that} \]
\[
\vartheta(z - \alpha x - \beta y) \leq \rho(z) - \alpha \rho(x) - \beta \rho(y).
\]

In the scalar-valued case \( F = \mathbb{R} \), our sandwich theorem 1.2 reduces to the main result of Fuchssteiner and König [5]. However, our proof for the vector-valued case from [24] is substantially different from the approach given in [5] and [16], which we found hard to mimick. In the vector-valued setting, certain special cases of theorem 1.2 have also been obtained by Martellotti and Salvadori in [19], but again their techniques are entirely different from ours. In [24] we deduce the sandwich theorem 1.2 from the vector-valued version of the Mazur-Orlicz theorem 1.1 and the following simple approximation result. We include this result to shed some new light on the weak convexity conditions (1)-(3).

1.3. Approximation Lemma. Let \( T \) be a family of upper bounded mappings from an arbitrary non-empty set \( X \) into a Dedekind complete vector lattice \( F \). Moreover, assume that for some real number \( 0 < \lambda < 1 \) the following condition is fulfilled:
For all \( f, g \in T \) there exists some \( h \in T \) such that \( h \leq \lambda f + (1 - \lambda)g \) on \( X \).

Then we have \( \inf \{ \sup(f) : f \in T \} = \inf \{ \sup(f) : f \in \text{co}T \} \), where \( \text{co}T \) denotes, as usual, the convex hull of \( T \).

We note the following useful consequences of theorem 1.2, which improve corresponding results from [5] and [19] and have similar applications. As before, let \( E \) be an arbitrary real vector space, and let \( F \) denote a Dedekind complete vector lattice.

1.4. Cone Version. Let \( \vartheta : E \to F \) be a sublinear operator, and let \( K \) be a non-empty subset of \( E \) such that \( \vartheta \geq 0 \) on \( K \). Moreover, assume that for some pair of real numbers \( \alpha, \beta > 0 \) and some \( u \in F_+ \) the following condition is fulfilled:

For all \( x, y \in K \) and \( \varepsilon > 0 \) there is some \( z \in K \) such that \( \vartheta(z - \alpha x - \beta y) \leq \varepsilon u \).

Then there exists a linear operator \( \varphi : E \to F \) such that \( \varphi \geq 0 \) on \( K \) and \( \varphi \leq \vartheta \) on \( E \).

1.5. Convex Version. Let \( \vartheta : E \to F \) be a sublinear operator, and consider a subset \( K \) of \( E \) such that for some \( 0 < \lambda < 1 \) and some \( u \in F_+ \) the following condition is fulfilled:

For all \( x, y \in K \) and \( \varepsilon > 0 \) there is some \( z \in K \) such that \( \vartheta(z - \lambda x - (1 - \lambda)y) \leq \varepsilon u \).

Then there exists a linear operator \( \varphi : E \to F \) such that \( \varphi \leq \vartheta \) on \( E \) and

\[
\inf \{ \varphi(x) : x \in K \} = \inf \{ \vartheta(x) : x \in K \}.
\]

Proofs. Of course, 1.4 follows immediately from 1.2 with the choice \( \rho = 0 \). To prove 1.5, let us consider \( \rho := \inf \{ \vartheta(x) : x \in K \} \in F \cup \{-\infty\} \). If \( \rho = -\infty \), then every linear operator \( \varphi : E \to F \) with \( \varphi \leq \vartheta \) on \( E \) has the desired property whereas, in the case \( \rho \in F \), the sandwich theorem 1.2 can be applied to the constant function given by \( \rho \).

The last result is particularly useful in optimization theory and dates back, in the scalar-valued setting \( F = \mathbb{R} \), to König [11], [12]. The weak form of the convexity conditions in the preceding results is crucial for a number of applications. As already pointed out in [16] and [24], this type of convexity condition arises naturally in the theory of function algebras and in Choquet theory [12], [23]; in potential theory [11]; in the theory of monotone operators [12], [16]; in optimization theory [27]; and in the general theory of minimax theorems [5], [11], [16], [19], [21].

For the sake of illustration, we conclude this section with a short review of certain minimax results from [5] and [16]. Theorems of this particular type date back to Ky Fan, but it was König who discovered the close connection to certain extended versions of the Hahn-Banach theorem [11]. Given a family \( T \) of upper semicontinuous functions from a compact Hausdorff space \( S \) into the interval \( [-\infty, \infty] \), we are interested in conditions on \( T \) which imply the so-called minimax identity:

4
\[ (4) \quad \inf_{f \in T} \max_{x \in S} f(x) = \max_{x \in S} \inf_{f \in T} f(x). \]

Notice that the inequality "\( \geq \)" is always satisfied. Moreover, a simple reformulation of the classical Dini theorem shows that (4) holds whenever \( T \) is directed below. However, this condition is frequently too strong, and the following weak convexity conditions are sometimes more appropriate. For \( \alpha, \beta, \sigma, \tau > 0 \) we consider the following conditions:

\[
\begin{align*}
\text{Ext}(\alpha, \beta) & : f, g \in T, \varepsilon > 0 \Rightarrow \exists h \in T \text{ such that } h(x) \leq \alpha f(x) + \beta g(x) + \varepsilon \quad \forall x \in S \\
\text{Int}(\sigma, \tau) & : u, v \in S, \varepsilon > 0 \Rightarrow \exists w \in S \text{ such that } \sigma f(u) + \tau f(v) \leq f(w) + \varepsilon \quad \forall f \in T
\end{align*}
\]

Each of the following results 1.6 and 1.7 can be easily deduced from the sandwich theorem 1.2 or its corollary 1.4. For details and further information we refer to [5], [16], [21].

1.6. Abstract Average Theorem. Assume that \( \text{Ext}(\alpha, \beta) \) holds for a pair of real numbers \( \alpha, \beta > 0 \). If \( \sup f \geq 0 \) for all \( f \in T \), then there exists a regular Borel probability measure \( \varphi \in \text{Prob}(S) \) such that \( \int f \, d\varphi \geq 0 \) for all \( f \in T \).

1.7. Abstract Barycenter Theorem. Assume that \( \text{Int}(\sigma, \tau) \) holds for a pair of real numbers \( \sigma, \tau > 0 \). If there is some \( \varphi \in \text{Prob}(S) \) such that \( \int f \, d\varphi \geq 0 \) for all \( f \in T \), then there exists some \( x \in S \) such that \( f(x) \geq 0 \) for all \( f \in T \).

1.8. Combination and Minimax Theorem. Assume that both \( \text{Ext}(\alpha, \beta) \) and \( \text{Int}(\sigma, \tau) \) are fulfilled for certain \( \alpha, \beta, \sigma, \tau > 0 \). If \( \sup f \geq 0 \) for all \( f \in T \), then there exists some \( x \in S \) such that \( f(x) \geq 0 \) for all \( f \in T \). In particular, if \( \alpha + \beta = 1 \) and \( \sigma + \tau = 1 \), then the minimax identity (4) holds.

It is interesting to observe that the minimax theorem 1.8 ultimately follows from a double application of the Mazur-Orlicz theorem. Of course, 1.8 contains the classical minimax theorem due to John von Neumann as a special case. Let us also note that both 1.6 and 1.7 are of independent interest. For instance, theorem 1.7 immediately implies the existence of barycenters in the context of certain families of upper semicontinuous functions [21]. Finally, we should mention the interesting recent progress on minimax type theorems due to König, Kindler, and Simons, where conditions involving, for instance, connectedness replace the weak convexity conditions of the preceding result; see [33], [34], and further references given there.

2. Decomposition Theorems. Some important contributions to the Mazur-Orlicz theory have been made by König in a sequence of papers [11], [12], [13], [14]. One of his goals was to develop a general theory of sublinear functionals as a basic tool for problems in analysis. For the sake of illustration and popularization, we include the following series of elementary results from [13]. In the following, let \( K \) be a non-empty subset of a real vector space \( E \) such that \( \alpha K + \beta K \subseteq K \) holds for some pair of real numbers \( \alpha, \beta > 0 \), and consider finitely many sublinear functionals \( \vartheta_1, \ldots, \vartheta_r : E \to \mathbb{R} \).
2.1. Minimum Theorem. There exists a linear functional \( \varphi : E \to \mathbb{R} \) with the property \( \varphi \leq \min(\vartheta_1, \ldots, \vartheta_r) \) on \( E \) if and only if \( \vartheta_1(u_1) + \ldots + \vartheta_r(u_r) \geq 0 \) holds for all \( u_1, \ldots, u_r \in E \) with \( u_1 + \ldots + u_r = 0 \).

2.2. Maximum Theorem. A linear functional \( \varphi : E \to \mathbb{R} \) satisfies the condition \( \varphi \leq \max(\vartheta_1, \ldots, \vartheta_r) \) on \( K \) if and only if there exist real numbers \( \sigma_1, \ldots, \sigma_r \geq 0 \) with \( \sigma_1 + \ldots + \sigma_r = 1 \) such that \( \varphi \leq \sigma_1 \vartheta_1 + \ldots + \sigma_r \vartheta_r \) on \( K \).

2.3. Sum Theorem. A linear functional \( \varphi : E \to \mathbb{R} \) satisfies \( \varphi \leq \vartheta_1 + \ldots + \vartheta_r \) on \( K \) if and only if there exist linear functionals \( \varphi_1, \ldots, \varphi_r : E \to \mathbb{R} \) such that \( \varphi_k \leq \vartheta_k \) on \( E \) for \( k = 1, \ldots, r \) and \( \varphi \leq \varphi_1 + \ldots + \varphi_r \) on \( K \).

All these results can be easily deduced from the sandwich theorem 1.2 in the scalar-valued setting \( F = \mathbb{R} \); for details and further information we refer to [13] and [16]. Let us note that the sum theorem is certainly folklore in functional analysis: versions of this result can be found, for instance, in early work of Choquet, Klee, and others. However, we believe that the more thorough investigation of sublinear functionals and their systematic application in different areas of analysis originated from the work of König.

It is natural to ask for extensions of the previous results for certain infinite families of sublinear functionals. Elementary examples show that the maximum theorem does not extend even to the case of countably many sublinear functionals [13]. Actually, it turns out that certain continuous versions of the maximim theorem are closely related to general versions of the integral representation theorems due to Riesz, Choquet, and Bishop-de Leeuw; a thorough discussion of this aspect can be found in [6].

On the other hand, it is possible to extend the sum theorem to the case of countably many sublinear functionals and even to a more general measure-theoretic setting. Indeed, we have the following generalization of a disintegration theorem due to Strassen, which has been obtained in [22]. Obviously, the result contains the sum theorem as a special case.

2.4. Strassen Type Disintegration Theorem. Let \( (S, \Sigma, \sigma) \) be an arbitrary positive measure space, and consider a sublinear operator \( \vartheta : E \to L^1(\sigma) \) and a linear functional \( \varphi : E \to \mathbb{R} \) such that \( \varphi(x) \leq \int \vartheta(x) d\sigma \) for all \( x \in K \). Then there exists a linear operator \( T : E \to L^1(\sigma) \) such that \( \varphi(x) \leq \int T(x) d\sigma \) for all \( x \in K \) and \( T(x) \leq \vartheta(x) \) for all \( x \in E \).

The proof of this theorem given in [22] is based on a suitable combination of the theorems of Mazur-Orlicz and Radon-Nikodym. Actually, our method is general enough to carry over to a vector-valued setting: the theorem remains valid for sublinear operators \( \vartheta \) mapping \( E \) into the space \( L^1(\sigma, F) \) of Bochner integrable functions from \( S \) into \( F \), where \( F \) denotes a Dedekind complete Banach lattice with the Radon-Nikodym property. Finally, let us point out that the inequalities \( T(x) \leq \vartheta(x) \) from the preceding theorem are just for equivalence classes of integrable functions modulo the underlying measure \( \sigma \). Thus, in certain classical applications, where one is mainly interested in pointwise inequalities of functions, the disintegration theorem 2.4 has to be combined with lifting theory or suitable continuity and separability assumptions. For details we refer to [22].
3. Some Typical Sublinear Functionals. The results of the first section have obvious applications in traditional functional analysis. Indeed, the standard versions of the Hahn-Banach extension and separation theorems can be easily obtained from the Mazur-Orlicz theory by choosing the sublinear functional as a norm, semi-norm, or the Minkowski functional of an absorbing convex set. In the following, we shall present some further examples of sublinear functionals which we found particularly useful.

3.1. Measure Theory. First, let $E$ denote a linear subspace of the space $B(S)$ of all bounded real-valued functions on an arbitrary non-empty set $S$, and assume that $E$ contains the real constants. In addition to the usual supremum norm $\| \cdot \|_\infty$ on $E$, we consider the supremum functional $\vartheta = \sup$ given by $\vartheta(f) := \sup\{f(x) : x \in S\}$ for all $f \in E$. Then, for each linear functional $\varphi : E \to \mathbb{R}$, we have the following equivalences:

$$\varphi \leq \sup \iff \varphi \leq \| \cdot \|_\infty \text{ and } \varphi(1) = 1 \iff \varphi \text{ is positive and } \varphi(1) = 1$$

Indeed, if the linear functional $\varphi$ is dominated by $\| \cdot \|_\infty$ on $E$ and satisfies $\varphi(1) = 1$, then for each $f \in E$ with $f \geq 0$ we obtain the estimates

$$\frac{1}{2}\|f\|_\infty - \varphi(f) = \varphi\left(\frac{1}{2}\|f\|_\infty - f\right) \leq \frac{1}{2}\|f\|_\infty - f \leq \frac{1}{2}\|f\|_\infty$$

and therefore $\varphi(f) \geq 0$, which shows that the functional $\varphi$ is positive. The remaining implications are obvious.

Now, if $S$ is a compact Hausdorff space, we may apply the preceding result to the space $E = C(S)$ of all continuous real-valued functions on $S$. An obvious combination with the Riesz representation theorem leads to the following useful characterization: a linear functional $\varphi$ on $C(S)$ is dominated by the supremum functional $\vartheta$ if and only if there exists a regular Borel probability measure $\sigma$ on $S$ such that the representation $\varphi(f) = \int f d\sigma$ holds for all $f \in C(S)$.

More generally, given a closed subset $K$ of the compact Hausdorff space $S$, let us consider the sublinear functional $\vartheta$ on $C(S)$, given by $\vartheta(f) := \sup\{f(x) : x \in K\}$ for all $f \in C(S)$. Then it is easily seen that the linear functionals $\varphi$ on $C(S)$ dominated by this functional $\vartheta$ correspond, via integration, to the regular Borel probability measures on $S$ supported by $K$. Thus, an application of the Mazur-Orlicz theory in this context will yield the existence of probability measures with certain additional properties. This approach leads to very short and natural proofs, for instance, in Choquet theory and the theory of function algebras. Various examples can be found in [12], [16], [23].

3.2. Abstract Hardy Algebras. Let $(S, \Sigma, m)$ be a probability measure space, and consider a weak * closed complex subalgebra $H$ of $L^\infty(m)$, which contains the complex constants. Our basic assumption is that integration is multiplicative on $H$ in the sense of $\int uvdm = \int udm \cdot \int vdm$ for all $f \in H$. This is the abstract Hardy algebra situation, which has been thoroughly investigated by Barbey and König in [2]. Of course, the classical example is the Hardy algebra $H = H^\infty$ of all nontangential limits of bounded analytic functions.
functions on the unit disc, where \( m \) denotes the normalized Lebesgue measure on the unit circle \( S \). In the general theory of Hardy algebras, an important role is played by the set of representing densities

\[
M := \{ V \in L^1(m) : V \geq 0 \text{ and } \int u dm = \int uV dm \text{ for all } u \in H \}.
\]

Obviously, the set \( M \) is convex, bounded, and weakly closed. The general idea is that the abstract theory for \( H \) will be very similar to the classical \( H^\infty \)-theory whenever \( M \) is small in a suitable sense. The strongest results are possible in the so-called Szegö situation \( M = \{1\} \). However, a surprisingly large number of results can also be obtained under the assumption that \( M \) is weakly compact. This situation arises, for instance, in the theory of rational approximation on compact subsets of the plane with only finitely many holes. To deal with the weak compactness of \( M \), the following functional turns out to be useful. Let \( E \) denote the space of all real-valued functions in \( L^\infty(m) \) and consider

\[
\vartheta(f) := \inf \{ Re \int u dm : u \in H \text{ with } Re u \geq f \} \text{ for all } f \in E.
\]

Then it can be shown that \( \vartheta : E \to \mathbb{R} \) is sublinear and satisfies

\[
\vartheta(f) = \sup \{ \int f V dm : V \in M \} \text{ for all } f \in E.
\]

From this identity it is not hard to conclude the following result: the set \( M \) is weakly compact if and only if for every linear functional \( \varphi \) on \( E \) with \( \varphi \leq \vartheta \) there exists some \( V \in M \) such that \( \varphi \) is of the form \( \varphi(f) = \int fV dm \) for all \( f \in E \). We refer to proposition IV.4.5 of [2] for a proof of this characterization of weak compactness, to chapter VIII of [2] for various applications, and to [12] for some related results.

### 3.3. Submodular Set Functions

In this last example, we are concerned with a useful sublinear functional given by integration with respect to a submodular set function. Let \( S \) be an algebra of subsets of a given non-empty set \( \Sigma \), and consider a set function \( \nu : \Sigma \to \mathbb{R} \), which is submodular in the sense of

\[
\nu(\emptyset) = 0 \quad \text{and} \quad \nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B) \quad \text{for all } A, B \in \Sigma.
\]

Furthermore, let \( E \) denote the space of all \( \Sigma \)-measurable simple functions \( f : S \to \mathbb{R} \). Then, for each \( f \in E \), we introduce the integral \( \vartheta(f) \) as follows:

\[
(5) \quad \vartheta(f) := \int_0^\infty \nu([g \geq t]) dt + c \nu(S) \quad \text{for all } g \in E_+ \text{ and } c \in \mathbb{R} \text{ with } f = g + c.
\]

It is not hard to see that (5) yields a well-defined real number \( \vartheta(f) \), which coincides with the elementary integral \( \int f d\nu \) whenever \( \nu \) is an additive set function. Moreover, it can be shown that \( \vartheta : E \to \mathbb{R} \) is a sublinear functional. This useful fact seems to date back to
Choquet [3] in a slightly different situation. More elementary proofs of the sublinearity of \( \vartheta \) can be found in [10] and [25].

Using the method of sublinear integration and a suitable sandwich theorem, Kindler [10] could establish a Mazur-Orlicz type theorem for submodular set functions. In the same vein, we could prove the following result [25]: for every submodular set function \( \nu : \Sigma \to \mathbb{R} \) and every chain \( \Gamma \) in \( \Sigma \), there exists an additive set function \( \mu : \Sigma \to \mathbb{R} \) such that \( \mu \leq \nu \) on \( \Sigma \) and \( \mu = \nu \) on \( \Gamma \). This observation has interesting applications, for instance, in network theory where submodular set functions arise naturally in connection with capacities. For details we refer to [25].

4. Generalized Convex Operators. Throughout this section, let \( E \) be a real vector space, and let \( F \) denote a Dedekind complete ordered real vector space. It will be convenient to add a smallest element \(-\infty\) to the space \( F \) and to extend the algebraic operations from \( F \) to \( F_* := F \cup \{-\infty\} \) in the usual way, i.e. we define \( 0 \cdot (-\infty) := 0 \), \( t \cdot (-\infty) := -\infty \) for all real \( t > 0 \), and \( x + (-\infty) := -\infty \) for all \( x \in F \).

Now, given a subset \( \Lambda \) of the unit interval \([0,1]\), a subset \( K \) of \( E \) is said to be \( \Lambda \)-convex if \( \lambda u + (1-\lambda)v \in K \) holds for all \( u, v \in K \) and all \( \lambda \in \Lambda \). Moreover, if \( K \) is a \( \Lambda \)-convex subset of \( E \), then an operator \( T : K \to F_* \) is called \( \Lambda \)-convex if \( T(\lambda u + (1-\lambda)v) \leq \lambda T(u) + (1-\lambda)T(v) \) holds for all \( u, v \in K \) and \( \lambda \in \Lambda \). If these inequalities hold with "\( \geq \)" instead of "\( \leq \)" for all \( u, v \in K \) and \( \lambda \in \Lambda \), then the operator \( T : K \to F_* \) is called \( \Lambda \)-concave, resp. \( \Lambda \)-affine. Of course, all these notions are of interest only when \( \Lambda \cap (0,1) \) is non-empty. Finally, we use the pointwise order relation for operators from \( K \) into the space \( F_* : \) for \( S, T : K \to F_* \) we write \( S \leq T \) on \( K \) if and only if \( S(x) \leq T(x) \) holds for all \( x \in K \).

In the following, we shall extend various well-known results from convex analysis to the more general case of \( \Lambda \)-convex operators. Our approach seems to be new even in the classical setting of convex functions on a convex set.

4.1. Support Theorem. Let \( K \subseteq E \) be \( \Lambda \)-convex for some given subset \( \Lambda \) of \([0,1]\), and consider a \( \Lambda \)-convex operator \( T : K \to F_* \). Then for every \( u \in K \) there exists a \( \Lambda \)-affine operator \( S : K \to F_* \) such that \( S \leq T \) on \( K \) and \( S(u) = T(u) \).

Proof. Without loss of generality, we may assume that \( T(u) > -\infty \) and that \( \emptyset \neq \Lambda \subseteq (0,1) \). Let \( \Pi \) be the set of all \( \Lambda \)-convex operators \( S : K \to F_* \) which satisfy \( S \leq T \) on \( K \) and \( S(u) = T(u) \). Then \( \Pi \) is non-empty and downward inductive in the pointwise order relation, since for every chain \( \Gamma \) in \( \Pi \) the pointwise infimum over \( \Gamma \) exists, by the Dedekind completeness of \( F \), and certainly defines a lower bound for \( \Gamma \) in \( \Pi \). Hence, by Zorn’s lemma, there exist minimal members of \( \Pi \), and the proof will be completed by showing that the minimal members of \( \Pi \) are actually \( \Lambda \)-affine. Now, given a minimal \( S \in \Pi \) and an arbitrary \( \lambda \in \Lambda \), let us introduce the operator \( P : K \to F_* \) by

\[
P(x) := \frac{1}{\lambda} S(\lambda x + (1-\lambda)u) - \frac{1-\lambda}{\lambda} S(u) \text{ for all } x \in K.
\]
It is easily seen that $P$ is again $\Lambda$-convex and satisfies $P \leq S$ on $K$ as well as $P(u) = S(u) = T(u)$. Since $S$ is minimal in $\Pi$, we conclude that $P = S$ on $K$ and consequently

$$S(\lambda x + (1 - \lambda)u) = \lambda S(x) + (1 - \lambda)S(u) \text{ for all } x \in K.$$  

To finish the proof, we have to repeat the preceding argument for a slightly modified operator. Let us fix an arbitrary $x \in K$ with $S(x) > -\infty$ and define

$$Q(y) := \frac{1}{1-\lambda}S(\lambda x + (1-\lambda)y) - \frac{\lambda}{1-\lambda}S(x) \text{ for all } y \in K.$$  

As before, it is immediate that the operator $Q : K \to F_\ast$ is $\Lambda$-convex and satisfies $Q \leq S$ on $K$. And from (6) it is clear that $Q(u) = S(u) = T(u)$. Again, it follows from the minimality of $S$ in $\Pi$ that $Q = S$ on $K$ and therefore

$$S(\lambda x + (1 - \lambda)y) = \lambda S(x) + (1 - \lambda)S(y) \text{ for all } y \in K.$$  

Let us finally observe that for $x \in K$ with $S(x) = -\infty$ the last identity follows immediately from the $\Lambda$-convexity of $S$. Thus $S$ is indeed $\Lambda$-affine, which completes the proof of 4.1.

Now, given an arbitrary non-empty subset $\Lambda$ of $[0,1]$, let $[\Lambda]$ denote the intersection of $[0,1]$ with the subfield generated by $\Lambda$ in $\mathbb{R}$. Thus $[\Lambda]$ contains the rationals $\mathbb{Q} \cap [0,1]$ and hence is dense in $[0,1]$. We shall show that, for a generalized convex operator defined on a convex set, the convexity with respect to $\Lambda$ automatically extends to $[\Lambda]$ as soon as $\Lambda \cap (0,1)$ is known to be non-empty. In particular, it turns out that non-trivial generalized convex operators are always rationally convex. The following result has been obtained by Kuhn [18] in the scalar-valued case $F = \mathbb{R}$. Let us point out, however, that the approach presented here is not only more general, but also much more direct and self-contained, since we do not have to invoke the somewhat sophisticated abstract version of the Hahn-Banach theorem due to to Rodé [30].

4.2. Permanence Theorem. Let $\Lambda$ be a subset of $[0,1]$, and consider an operator $T : K \to F_\ast$ on a convex subset $K$ of $E$. Then we have:

(a) If $T$ is $\Lambda$-affine and $\Lambda \cap (0,1)$ is non-empty, then $T$ is $[\Lambda]$-affine.
(b) If $T$ is $\Lambda$-convex and $\Lambda \cap (0,1)$ is non-empty, then $T$ is $[\Lambda]$-convex.

Note that assertion (a) may be considered as a special case of assertion (b), since the $\Lambda$-affine and the $\Lambda$-convex operators from $K$ into $F_\ast$ certainly coincide, if $F$ is endowed with the trivial Dedekind complete order given by equality "=". But, actually, assertion (a) is an elementary fact, which has to be combined with the support theorem 4.1 to establish (b).

Proof. To prove assertion (a), let us define $A(T)$ to be the set of all $\alpha \in [0,1]$ for which the identity $T(\alpha u + (1-\alpha)v) = \alpha T(u) + (1-\alpha)T(v)$ holds for all $u, v \in K$. Then it is
clear that $\Lambda \cup \{0, 1\} \subseteq A(T)$, and the following series of arguments will show that actually $[\Lambda] \subseteq A(T)$. First, it is easily seen that $\alpha \beta + (1 - \alpha)\gamma \in A(T)$ for all $\alpha, \beta, \gamma \in A(T)$. In particular, we obtain that

$$(7) \quad \alpha, \beta \in A(T) \text{ implies } 1 - \alpha \in A(T) \text{ and } \alpha \beta \in A(T).$$

Now the crucial step is to show that

$$(8) \quad \alpha, \beta \in A(T) \text{ with } \alpha \leq \beta \text{ and } \beta \neq 0 \implies \frac{\alpha}{\beta} \in A(T).$$

To prove this assertion, we may assume $\alpha < \beta$ so that $\gamma := \alpha/\beta \in [0, 1)$. Given arbitrary $u, v \in K$, let $w := \gamma u + (1 - \gamma)v$. Then $w \in K$ satisfies $\beta w + (1 - \beta)v = \alpha u + (1 - \alpha)v$ and hence $\beta T(w) + (1 - \beta)T(v) = \alpha T(u) + (1 - \alpha)T(v)$. We conclude that

$$T(w) + \left(\frac{1}{\beta} - 1\right)T(v) = \gamma T(u) + \left(\frac{1}{\beta} - \gamma\right)T(v),$$

which implies the desired identity $T(w) = \gamma T(u) + (1 - \gamma)T(v)$ provided that $T(v) \in F$. In the remaining case $T(v) = -\infty$ we have to show that also $T(w) = -\infty$. To this end, let us fix some $\lambda \in \Lambda \cap (0, 1) \subseteq A(T)$ and choose an integer $k \in \mathbb{N}$ such that $\lambda^k < 1 - \gamma$. From (7) we know that $\mu := 1 - \lambda^k \in A(T)$. Since

$$w = \gamma u + (1 - \gamma)v = \mu z + (1 - \mu)v \text{ with } z := \frac{\gamma}{\mu} u + \left(1 - \frac{\gamma}{\mu}\right)v \in K,$$

we arrive at $T(w) = \mu T(z) + (1 - \mu)T(v)$, which shows that $T(v) = -\infty$ implies $T(w) = -\infty$. Thus $T(\gamma u + (1 - \gamma)v) = \gamma T(u) + (1 - \gamma)T(v)$ holds for all $u, v \in K$, which completes the proof of (8). We next observe that

$$(9) \quad \alpha, \beta \in A(T) \text{ with } \alpha \leq \beta \implies \beta - \alpha \in A(T).$$

Indeed, since $\beta - \alpha = \beta(1 - \alpha/\beta)$ if $\beta \neq 0$, this assertion is an immediate consequence of (7) and (8). Furthermore, it follows easily from (9) that

$$(10) \quad \alpha, \beta \in A(T) \text{ with } \alpha + \beta \leq 1 \implies \alpha + \beta \in A(T),$$

since $\beta \leq 1 - \alpha$ with $1 - \alpha \in A(T)$ implies $1 - \alpha - \beta \in A(T)$ by (9) and thus $\alpha + \beta \in A(T)$. We finally claim that the set of quotients $X := \{\pm \alpha/\beta : \alpha, \beta \in A(T) \text{ with } \beta > 0\}$ satisfies

$$(11) \quad \frac{\alpha}{\beta} \pm \frac{\gamma}{\delta} \in X \text{ for all } \alpha, \beta, \gamma, \delta \in A(T) \text{ with } \beta, \delta > 0.$$

To prove this assertion, we fix some $\lambda \in \Lambda \cap (0, 1) \subseteq A(T)$ and choose an integer $k \in \mathbb{N}$ such that $\mu := \lambda^k$ satisfies $|\alpha \delta \pm \beta \gamma|/\mu < 1$. Then it is clear from (7), (9) and (10) that $\alpha \delta \mu \pm \beta \gamma \mu \in \pm A(T)$ and therefore

$$\frac{\alpha}{\beta} \pm \frac{\gamma}{\delta} = \frac{\alpha \delta \mu \pm \beta \gamma \mu}{\beta \delta \mu} \in X.$$
Now an obvious combination of (7) and (11) shows that $X$ is a subfield of $\mathbb{R}$, from which we conclude that $[\Lambda] \subseteq [A(T)] \subseteq X \cap [0,1]$. On the other hand, another application of (8) reveals that $X \cap [0,1] \subseteq A(T)$. It follows that $[\Lambda] \subseteq A(T)$, which completes the proof of assertion (a). To prove part (b), let $x, y \in K$ and $\mu \in [\Lambda]$ be arbitrarily given and consider $u := \mu x + (1 - \mu)y \in K$. By the support theorem 4.1, there exists a $\Lambda$-affine operator $S : K \to F_*$ such that $S \leq T$ on $K$ and $S(u) = T(u)$. From (a) we conclude that $u \in [\Lambda] \subseteq A(S)$ and therefore $T(\mu x + (1 - \mu)y) = T(u) = S(u) = \mu S(x) + (1 - \mu)S(y) \leq \mu T(x) + (1 - \mu)T(y)$, which proves that $T$ is indeed $[\Lambda]$-convex. The assertion follows.

It is interesting to observe that theorem 1.2 is sharp even in the scalar-valued setting $F = \mathbb{R}$. Indeed, it has been shown by Ger [7] that, for every convex set $K$ consisting of more than one point and for every subfield $X$ of $\mathbb{R}$, there exists a function $T : K \to X$ with the following properties: the largest subset $\Lambda$ of $[0,1]$ for which $T$ is $\Lambda$-affine is precisely $X \cap [0,1]$, and the largest subset $\Lambda$ of $[0,1]$ for which $T$ is $\Lambda$-convex is again precisely $X \cap [0,1]$. Actually, Ger was only interested in the case of $\Lambda$-convexity, but it is clear from the proof of theorem 1 in [7] that his construction yields a function $T : K \to X$ which is even affine with respect to $X \cap [0,1]$.

We close this section with the following version of Jensen’s inequality for generalized convex operators. Let us note that the result contains the case of $\Lambda$-affine operators, since we may endow the range space $F$ with the trivial order structure given by equality “$=$”.

4.3. Jensen Type Inequality. Again, let $\Lambda$ be a subset of $[0,1]$, and consider a $\Lambda$-convex operator $T : K \to F_*$ on a convex subset $K$ of $E$. Then we have

$$T\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i T(x_i) \quad \text{for all} \quad x_1, \ldots, x_n \in K \quad \text{and} \quad \lambda_1, \ldots, \lambda_n \in \Lambda \quad \text{with} \quad \sum_{i=1}^{n} \lambda_i = 1.$$

The proof follows immediately from part (b) of theorem 4.2 by an obvious induction argument. Although the result looks quite elementary, we do not know of any proof of this inequality, which avoids making use of a result like the support theorem 4.1. We finally note the following generalization of the support theorem. The proof is slightly more involved than the proof of 4.1, but follows the same line of argument. The result can be used to give an alternative transparent proof of the Mazur-Orlicz type theorem 1.2 and hence implies the main results of [5] and [19]. For details and further information we refer to our recent paper [26].

4.4. Sandwich Theorem for Generalized Convex Operators. Let $K$ be a $\Lambda$-convex subset of $E$ for some given subset $\Lambda$ of $[0,1]$, and consider a $\Lambda$-convex operator $T : K \to F_*$ and a $\Lambda$-concave operator $R : K \to F_*$ such that $R \leq T$ on $K$. Then there exists a $\Lambda$-affine operator $S : K \to F_*$ such that $R \leq S \leq T$ on $K$.

Proof. If $R$ is constant $= -\infty$, the assertion follows with the obvious choice $S := -\infty$. Hence we may assume that the set $X := \{u \in K : R(u) > -\infty\}$ is non-empty so that $C := \inf\{T(u) - R(u) : u \in X\}$ exists in $F$ and satisfies $C \geq 0$. Now, somewhat similar
to the proof of 4.1, let $\Pi$ consist of all those $\Lambda$-convex operators $S : K \to F_*$ which satisfy $R \leq S \leq T - C$ on $K$. Note that $\Pi$ is non-empty, since $T - C$ belongs to $\Pi$, and that $\Pi$ is downward inductive in the pointwise order relation, since $F$ is Dedekind complete. Thus Zorn’s lemma yields the existence of minimal operators in $\Pi$, and it remains to show that the minimal operators in $\Pi$ are actually $\Lambda$-affine. Hence, let $S \in \Pi$ be minimal, and consider an arbitrary $\lambda \in \Lambda$ with $0 < \lambda < 1$. Then the operator $P : K \to F_*$ given by

$$P(x) := \inf \left\{ \frac{1}{\lambda} S(\lambda x + (1 - \lambda)u) - \frac{1 - \lambda}{\lambda} R(u) : u \in X \right\} \text{ for all } x \in K$$

obviously satisfies $R \leq P$ on $K$, and a simple calculation based on the $\Lambda$-convexity of $S$ on $K$ and the $\Lambda$-concavity of $R$ on $X$ reveals that $P$ is $\Lambda$-convex. Finally, from the estimates

$$\frac{1}{\lambda} S(\lambda x + (1 - \lambda)u) - \frac{1 - \lambda}{\lambda} R(u) \leq S(x) + \frac{1 - \lambda}{\lambda} S(u) - \frac{1 - \lambda}{\lambda} R(u) \leq$$

$$\leq S(x) + \frac{1 - \lambda}{\lambda} [T(u) - R(u) - C]$$

for all $x \in K$ and $u \in X$ and from the definition of $C$, we infer that $P \leq S$ on $K$. Therefore, the minimality of $S$ in $\Pi$ implies $P = S$ on $K$ and hence

$$S(\lambda x + (1 - \lambda)u) - \lambda S(x) \geq (1 - \lambda)R(u) \text{ for all } u \in K \text{ and all } x \in X \text{ with } S(x) > -\infty.$$  

These inequalities will enable us to show that $S$ is $\lambda$-affine. Given an arbitrary $x \in X$ with $S(x) > -\infty$, let us introduce the operator $Q : K \to F_*$ by

$$Q(y) := \frac{1}{1 - \lambda} S(\lambda x + (1 - \lambda)y) - \frac{\lambda}{1 - \lambda} S(x) \text{ for all } y \in K.$$  

Then the preceding estimates guarantee that $R \leq Q$ on $K$. Moreover, it is easily verified that $Q$ is $\Lambda$-convex and satisfies $Q \leq S$ on $K$. Again, by the minimality of $S$ in $\Pi$, we conclude that $Q = S$ on $K$ and consequently $S(\lambda x + (1 - \lambda)y) = \lambda S(x) + (1 - \lambda)S(y)$ for all $y \in K$. Since this identity obviously holds for all $x \in X$ with $S(x) = -\infty$, we infer that $S$ is $\Lambda$-affine. The assertion follows.

4.5. Corollary. Let $K, L \subseteq E$ be $\Lambda$-convex for some $\Lambda \subseteq [0, 1]$ such that $0 \neq L \subseteq K$, and consider a $\Lambda$-convex operator $T : K \to F_*$. Then for every $\Lambda$-concave operator $R : L \to F_*$ with $R \leq T$ on $L$ there exists some $\Lambda$-affine operator $S : K \to F_*$ such that $R \leq S$ on $L$ and $S \leq T$ on $K$. In particular, for a suitable $\Lambda$-affine operator $S : K \to F_*$ with the property $S \leq T$ on $K$ we have $\inf\{S(u) : u \in L\} = \inf\{T(u) : u \in L\}$.

Proof. If we extend the operator $R$ from $L$ to $K$ by defining $R_*(x) := R(x)$ for all $x \in L$ and $R_*(x) := -\infty$ for all $x \in K \setminus L$, then $R_* : K \to F_*$ is $\Lambda$-concave and satisfies $R_* \leq T$ on $K$. Hence the first assertion is immediate after 4.4, whereas the final assertion follows by taking $R : L \to F_*$ to be constant $= \inf\{T(u) : u \in L\}$.
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Bilinear forms on $C^*$-algebras and invariant means

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In [10], Orlicz proved the following theorem, the inequality in which is often called Orlicz’s Inequality.

**Theorem 1** Let $\{a_{ij}\}$ be an infinite real matrix and $M$ be a positive constant such that whenever $\{t_i\}$, $\{s_i\}$ are real and of norm $\leq 1$ in $\ell_\infty$ and $N \geq 1$, we have

$$|\Sigma_{i,j=1}^N a_{ij} t_i s_j| \leq M.$$ 

Then there exists a universal constant $K$ such that

$$\left(\Sigma_i (\Sigma_j |a_{ij}|)^2\right)^{1/2} \leq KM.$$ 

This remarkable inequality is a forerunner of Grothendieck’s inequal-

1
ity and the $K$ of the theorem can be taken to be Grothendieck's constant ([9]). Recently, a $C^*$-algebra version of Grothendieck's ine-
quality – due to Pisier and Haagerup – has been playing an important role in operator algebra theory, and the object of this paper is to discuss some amenability consequences of the \textit{Grothendieck-
Pisier-Haagerup inequality} for $C^*$-algebras.

Let $A$ and $B$ be $C^*$-algebras. Then $Bil(A, B) = (A \hat{\otimes} B)^*$ is the space of bounded bilinear forms on $A \times B$. We sometimes write $Bil(A)$ in place of $Bil(A, A)$. Haagerup ([?]1) generalized results of Grothendieck and Pisier to show that if $V \in Bil(A, B)$, then there exist states $\phi_1$, $\phi_2$ on $A$ and $\psi_1$, $\psi_2$ on $B$ such that

$$|V(a, b)| \leq \|V\|(|\phi_1(a^*a) + \phi_2(aa^*)|^{1/2}(|\psi_1(b^*b) + \psi_2(bb^*)|^{1/2}$$ (1)

for all $a \in A, b \in B$.

In [8], it is shown that $Bil(A, B)$ is the sum of four natural subspaces which we will denote by $Bil_{ij}(A, B) (i, j \in \{1, 2\})$. The form $V \in Bil_{ij}(A, B)$ if there exist states $\phi$ on $A$, $\psi$ on $B$ and a constant $K \geq 0$ such that for all $a \in A, b \in B$:

$$|V(a, b)| \leq K\phi(a^*a)^{1/2}\psi(bb^*)^{1/2} (i = 1, j = 1)$$ (2)
\[
| V(a, b) | \leq K \phi(aa^*)^{1/2}\psi(bb^*)^{1/2} \quad (i = 1, j = 2) \quad (3)
\]
\[
| V(a, b) | \leq K \phi(a^*a)^{1/2}\psi(b^*b)^{1/2} \quad (i = 2, j = 1) \quad (4)
\]
\[
| V(a, b) | \leq K \phi(aa^*)^{1/2}\psi(b^*b)^{1/2} \quad (i = 2, j = 2) \quad (5)
\]

The subspace \( Bil_{22}(A, B) \) is the space of \textit{completely bounded} forms on \( A \times B \).

Kaijser and Sinclair ([8]) show that \( Bil(A, B) \) is the sum of the subspaces \( Bil_{ij}(A, B) \) by expressing \( V \in Bil(A, B) \) as the sum of four elements in the appropriate subspaces. The four elements are expressed in terms of the universal representations and antirepresentations of \( A, B \).

However, for the purposes of the calculations used in proving the results below, it is convenient to use a related expression for \( V \) involving only the universal representations of \( A \) and \( B \). (In particular, when \( A = B \), we can sit \( A \) on its universal representation Hilbert space and omit any reference to representations.) This related expression can be derived from the Kaijser-Sinclair expression by identifying the universal antirepresentation of \( A, B \) with the representation \( \overline{\pi}^0 \) of \( \overline{A}^0, \overline{B}^0 \), where \( \pi \) is the universal representation of \( A, B \) and we use the notation of [3, (2.2.8)]. However, we have
preferred to give below a simple, direct proof of the expression.

Let \((a, \xi) \to a\xi, \ (b, \eta) \to b\eta\) be the universal representations of \(A\) and \(B\) on Hilbert spaces \(\mathcal{H}, \mathcal{K}\). Let \(B(\mathcal{H}, \mathcal{K}), B(\mathcal{H}, \mathcal{K})\) be the Banach spaces of bounded linear (resp. conjugate linear) maps from \(\mathcal{H}\) to \(\mathcal{K}\). Note that \(\overline{B(\mathcal{H}, \mathcal{K})} = B(\mathcal{H}, \overline{\mathcal{K}}) = B(\overline{\mathcal{H}}, \mathcal{K})\) where \(\overline{\mathcal{H}}\) is the Hilbert space conjugate to \(\mathcal{K}\) ([3, (2.2.8)]).

**Proposition 1** Let \(V \in B(\mathcal{H}, \mathcal{K})\). Then there exist \(\xi_i \in \mathcal{H}, \eta_i \in \mathcal{K}\) \((i = 1, 2)\), \(T_{11}, T_{22} \in B(\mathcal{H}, \mathcal{K})\) and \(T_{12}, T_{21} \in B(\mathcal{H}, \mathcal{K})\) such that for all \(a, b \in A,\)

\[
V(a, b) = T_{11}a\xi_1.b^*\eta_1 + T_{12}a^*\xi_2.b^*\eta_1 + b\eta_2.T_{21}a\xi_1 + b\eta_2.T_{22}a^*\xi_2. \quad (6)
\]

Further, \(\|T_{ij}\| \leq \|V\|\).

**Proof.** Let \(\phi_i, \psi_i\) be as in (1), and \(\xi_i \in \mathcal{H}, \eta_i \in \mathcal{K}\), such that

\[
\phi_i(a) = a\xi_i, \quad \psi_i(b) = b\eta_i, \quad i = 1, 2
\]

for all \(a \in A, \ b \in B\). Let \(\mathcal{H}_\xi, \mathcal{K}_\eta\) be the subspaces of \(\mathcal{H} \oplus \overline{\mathcal{H}}, \mathcal{K} \oplus \overline{\mathcal{K}}\) respectively defined by:

\[
\mathcal{H}_\xi = \{a\xi_1 \oplus a^*\xi_2 : a \in A\}^{-} \quad (7)
\]

\[
\mathcal{K}_\eta = \{b\eta_1 \oplus b^*\eta_2 : b \in B\}^{-} \quad (8)
\]
From (1),

$$|V(a, b)| \leq \|V\| \left(\|a\xi_1\|^2 + \|a^*\xi_2\|^2\right)^{1/2} \left(\|b\eta_1\|^2 + \|b^*\eta_2\|^2\right)^{1/2}$$

(9)

and so we can define a continuous sesquilinear form $F$ form on $\mathcal{H}_\xi \times \mathcal{K}_\eta$ by:

$$F(a\xi_1 \oplus a^*\xi_2, b^*\eta_1 \oplus b\eta_2) = V(a, b).$$

(10)

Hence there exists a bounded, linear operator $T : \mathcal{H}_\xi \rightarrow \mathcal{K}_\eta$ such that $\|F\| = \|T\|$ and $F(\alpha, \beta) = T\alpha \cdot \beta$ for all $(\alpha, \beta) \in \mathcal{H}_\xi \times \mathcal{K}_\eta$. From (9), we have $\|F\| \leq \|V\|$. Extend $T$ without increase of norm to a bounded linear operator $T : \mathcal{H} \times \overline{\mathcal{H}} \rightarrow \mathcal{K} \times \overline{\mathcal{K}}$.

The maps $T_{ij}$ are then obtained by regarding $T$ as a matrix

$$ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} $$

and using (10). Clearly,

$$\|T_{ij}\| \leq \|T\| = \|F\| \leq \|V\|. \quad \square$$

It is easily checked that the bilinear form in (10) associated with $T_{ij}$ belongs to $Bil_{ij}(A, B)$, so that we also obtain another proof of the result of Kaijser and Sinclair referred to above.
We now briefly discuss two amenability theorems for a unital $C^*$-algebra $A$. The proofs, which use the above proposition, will appear elsewhere.

The space $Bil(A)$ is a dual Banach $A$-module with actions given by:

$$xV(a,b) = V(ax,b) \quad Vx(a,b) = V(xa,b)$$

Direct checking shows that each $Bil_AB$ is an invariant subspace of $Bil(A)$. A virtual diagonal for $A$ is an element $M$ of $(Bil(A))^*$ such that $xM = Mx$ for all $x \in A$ and $\pi^{**}(M) = 1$, where $\pi : A \hat{\otimes} A \to A$ is the multiplication map. The algebra $A$ is called amenable if there is a virtual diagonal for $A$. The algebra $A$ is called strongly amenable if there is a virtual diagonal in the weak* closure of the convex hull of the elements $(u^* \otimes u)^*$, where $u$ is in the unitary group of $A$.

Amenability first arose in the context of groups. Indeed, let $G$ be a locally compact group. The Banach space $L_\infty(G)$ is a Banach $G$-module with actions:
\[ xf(y) = f(yx) \quad fx(y) = f(xy). \]

A state \( m \) on \( L_\infty(G) \) is a RIM (right invariant mean) if \( m(xf) = m(f) \) for all \( x \in G, f \in L_\infty(G) \) (equivalently, \( mx = m \) for all \( x \in G \)).

There are many formulations of amenability. (See [11].) Of particular significance is the following. Here \( LUC(G) \) is the space of bounded functions \( f : G \to \mathbb{C} \) such that the map

\[ x \to xf \]

is norm continuous. The space \( LUC(G) \) is topological in character.

**Theorem 2** The following are equivalent:

1. \( G \) is amenable

2. There exists a RIM on \( L_\infty(G) \).

3. There exists a RIM on \( LUC(G) \).

Note that (2) is *Functional Analytic* in character, while (3) is *topological*. A result below shows that a similar situation holds for \( C^* \)-algebras.

Now let \( G \) be the unitary group of \( A \). (The group \( G \) is usually not locally compact in any reasonable topology.) The connection
between virtual diagonals on $A$ and RIM’s on $G$ is given by the map

$$\Gamma : Bil(A) \rightarrow \ell_\infty(G)$$

where:

$$\Gamma(V)(v) = V(v^*,v)$$

We define the following subspaces:

$$B(A) = \Gamma(Bil(A)) \quad \quad \quad B_{ij}(A) = \Gamma(Bil_{ij}(A))$$

The subspace $B_{22}(A)$ of $\ell_\infty(G)$ is unital and invariant.

The group $G$ can be shown to be a topological group with the
relative weak topology of $A$ and $LUC(G)$ is defined with respect to
that topology.

**Theorem 3** The following are equivalent:

1. $A$ is amenable;

2. there exists a RIM on $B_{22}(A)$;

3. there exists a RIM on $LUC(G)$.

The final result gives an invariant mean characterization of strong
amenability.

**Theorem 4** The algebra $A$ is strongly amenable if and only if there
exists a RIM on $B(A)$. 
References


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MAHARAM'S PROBLEM

JAMES W. ROBERTS

This paper is dedicated to the memory of Professor Wladyslaw Orlicz

0. INTRODUCTION.

The purpose of this paper is to introduce the reader to the fascinating (and fiendish) currently unsolved problem posed by D. Stone in 1947. It is intended that this paper should be accessible to as wide an audience as possible. With only a few exceptions, a knowledge of Royden's book and a bit of basic functional analysis should suffice. For instance, we do not define Boolean algebras (since they are defined in Royden's book) but we do outline the development of Stone spaces. The author apologizes in advance to the experts. Throughout the exposition we shall derive a number of equivalent formulations of Maharam's Problem. These will be phrased as questions so that a positive answer to one is the same as a positive answer to another. The paper begins with results that are pretty standard (though not necessarily in print somewhere). No serious attempt is made to credit first discovery. The end of the paper (particularly the last two sections) reflects the author's view of the problem. The development of the subject is one of the least economical developments possible. It is hoped that it is a natural development. As one last bit of advice, do not work on this problem unless you have tenure!

1. THE PROBLEM.

We begin by defining the notion of a submeasure and notions related to submeasures.

Definition 1.1. Suppose $\mathcal{A}$ is a Boolean algebra and $\nu : \mathcal{A} \rightarrow [0, \infty)$ such that (1) $\nu(0) = 0$.

(2) if $A, B \in \mathcal{A}$ and $A \leq B$ then $\nu(A) \leq \nu(B)$ ($\nu$ is monotone).

(3) if $A, B \in \mathcal{A}$ then $\nu(A \vee B) \leq \nu(A) + \nu(B)$ ($\nu$ is subadditive).

then $\nu$ is called a submeasure on $\mathcal{A}$.

Definition 1.2. If $\mathcal{A}$ is a $\sigma$-algebra of subsets of a set $\Omega$ and $\nu$ is a submeasure on $\mathcal{A}$, $\nu$ is called a continuous submeasure if whenever $\langle A_n \rangle$ is a decreasing sequence in $\mathcal{A}$ such that

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

then

$$\lim_{n \to \infty} \nu(A_n) = 0.$$
Examples. (1) Let $A$ be any algebra of subsets of some set $\Omega$ and let $B$ be a nonempty subset of $\Omega$. For $A \in A$ define
\[ \nu(A) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{if } A \cap B = \emptyset \end{cases} \]
\(\nu\) is a submeasure on $A$.

(2) Suppose, once again, that $A$ is an algebra of subsets of a set $\Omega$ and $\mu$ is a finitely additive vector measure taking values in an $F$-space $\langle X, \| \cdot \| \rangle$. Also suppose
\[ \sup \{ \| \mu(A) \| : A \in A \} < \infty \]
Let $\nu(A) = \sup \{ \| \mu(B) \| : B \subseteq A, B \in A \}$.

(3) Suppose $\phi$ is a nondecreasing subadditive function on $[0, \infty)$ taking nonnegative values such that
\[ \lim_{x \to 0^+} \phi(x) = 0. \]
If $(\Omega, A, \mu)$ is a finite measure space then $\nu = \phi \circ \mu$ is a continuous submeasure on $A$.

(4) Suppose $A$ is a $\sigma$-algebra of subsets of a set $\Omega$ and suppose that $\langle \mu_n \rangle$ is a sequence of finite measures on $A$ such that
\[ \lim_{n \to \infty} \mu_n(\Omega) = 0 \]
For $A \in A$, define $\nu(A) = \max_{n \in \mathbb{N}} \mu_n(A)$. Then $\nu$ is a continuous submeasure on $A$.

(5) Let $\nu$ denote Lebesgue outer measure on the power set of $[0, 1]$. $\nu$ is a submeasure.
Also if $\langle A_n \rangle$ is an increasing sequence of sets then
\[ \lim_{n \to \infty} \nu(A_n) = \nu(\bigcup_{n=1}^{\infty} A_n) \]
\(\nu\) is also countably subadditive. $\nu$ is, unfortunately, not a continuous submeasure. This fact is the content of problem (17b) in [9, p.66].

Definition 1.3. Suppose $\nu_1$ and $\nu_2$ are submeasures on a Boolean algebra $A$. We say that $\nu_2$ is absolutely continuous with respect to $\nu_1$ if for any sequence $\langle A_n \rangle$ in $A$
\[ \lim_{n \to \infty} \nu_1(A_n) = 0 \] implies that
\[ \lim_{n \to \infty} \nu_2(A_n) = 0. \]
$\nu_1$ is equivalent to $\nu_2$ if $\nu_1$ is absolutely continuous with respect to $\nu_2$ and $\nu_2$ is absolutely continuous with respect to $\nu_1$.

Notice that the above definition has an equivalent $\varepsilon - \delta$ formulation. As the plot unfolds we shall present a number of equivalent formulations of Maharam’s Problem. The following is the first of these.

Maharam’s Problem I:
Suppose $A$ is a $\sigma$-algebra of subsets of a set $X$. If $\nu$ is a continuous submeasure on $A$, does there exist a finite measure on $A$ equivalent to $\nu$?
2. SUBMEASURE ALGEBRAS.

Suppose \( \nu \) is a continuous submeasure on a \( \sigma \)-algebra \( \mathcal{A} \) of subsets of a set \( \Omega \). If \( A, B \in \mathcal{A} \), we say that \( A \) is equivalent to \( B \) if \( \nu(A \Delta B) = 0 \). This, of course, is an equivalence relation on \( \mathcal{A} \). We denote the equivalence class of \( A \) by \([A]\) and we let

\[
\mathcal{A} = \{ [A] : A \in \mathcal{A} \}
\]

Note that if we let \( D(A, B) = \nu(A \Delta B) \) for \( A, B \in \mathcal{A}, D \) is a pseudometric and \( \mathcal{A} \) is the corresponding metric space. Also \( \mathcal{A} \) is a Boolean algebra. We define for \([A],[B] \in \mathcal{A}\)

1. \([A] \lor [B] = [A \cup B]
2. \([A] \land [B] = [A \cap B]
3. 1 = [\Omega]
4. 0 = [\emptyset]
5. \([A]' = \overline{[A]}\)
6. \([A] \Delta [B] = [A \Delta B]\)

It is easily verified that these operations are well defined (independent of the representatives \( A, B \in \mathcal{A} \)) and that \( \mathcal{A} \) is a Boolean algebra. \( \mathcal{A} \) is called the submeasure algebra induced by \( (\nu, \mathcal{A}, \Omega) \).

It is customary to denote the members of a Boolean algebra with lower case letters. If \( a \) and \( b \) are members of a Boolean algebra we say that \( a \leq b \) if \( a \land b = a \). If \( \mathcal{C} \) is a nonempty collection in a Boolean algebra we say \( a \) is an upper bound for \( \mathcal{C} \) if \( b \leq a \) for every \( b \in \mathcal{C} \). We denote this by writing \( \mathcal{C} \leq a \). \( A \) is a least upper bound for \( \mathcal{C} \) if

1. \( \mathcal{C} \leq a \)
2. if \( \mathcal{C} \leq b \) then \( a \leq b \)

It is easily seen that any nonempty collection has at most one upper bound. If a collection \( \mathcal{C} \) has an upper bound, it is denoted \( \vee \mathcal{C} \). The notions of lower bound and greatest lower bound are defined similarly. The greatest lower bound of a collection \( \mathcal{C} \) is denoted by \( \wedge \mathcal{C} \). If every nonempty countable collection in a Boolean algebra has a least upper bound, we say that the Boolean algebra is \( \sigma \)-complete. If every nonempty collection has a least upper bound we say that the Boolean algebra is complete. We note that the submeasure algebra \( \mathcal{A} \) induced by \( (\nu, \mathcal{A}, \Omega) \) is \( \sigma \)-complete.

Definition 2.1.

A Boolean \( \sigma \)-algebra \( \mathcal{A} \) is called a submeasure algebra if there exists \( \nu \) a submeasure on \( \mathcal{A} \) such that

1. For every \( a \in \mathcal{A} \sim \{0\}, \nu(a) > 0 (\nu \text{ is strictly positive})
2. If \( \{a_n : n \in N\} \) is a sequence in \( \mathcal{A} \) such that \( a_1 \geq a_2 \cdots \) and \( \wedge \{a_n : n \in N\} = 0 \) then \( \lim_{n \to \infty} \nu(a_n) = 0 \) (\( \nu \) is a continuous submeasure).

Of course the submeasure algebra induced by a continuous submeasure on a \( \sigma \)-algebra is a submeasure algebra. As we shall see momentarily, the converse is true. Given an abstract Boolean algebra we need to get our hands on a concrete set \( \Omega \). The key idea here is the notion of a Stone space. Suppose \( \Omega \) is a compact Hausdorff space. We shall say that a set which is both closed and open is a clopen set and we shall denote the collection of all clopen sets by \( \text{Cl}(\Omega) \). Clearly \( \text{Cl}(\Omega) \) is an algebra of sets. If \( \text{Cl}(\Omega) \) is a base for the topology
of $\Omega$ (\textit{\Omega is totally disconnected}) $\Omega$ is called a \textit{Stone space}. The Cantor set is an example of a Stone space. If $S$ is any set, the product space $\{0, 1\}^S$ is also an example.

\textbf{The Stone Representation Theorem:}
If $A$ is a Boolean algebra, then there exists a Stone space $\Omega$ such that $A$ is a Boolean isomorphic to $cl(\Omega)$. Furthermore, if $\Omega_1$ and $\Omega_2$ are two such Stone spaces, then $\Omega_1$ is homeomorphic to $\Omega_2$.

Because Stone spaces representing a Boolean algebra $A$ are unique up to homeomorphism, we call any such Stone space the \textit{Stone space} of $A$. We now provide a sketch of how one obtains a Stone space. To motivate the idea, suppose that $\Omega$ is a Stone space for a Boolean algebra $A$ and $x \in \Omega$. For $E \in cl(\Omega)$,

$$\phi_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

defines a two valued measure on the algebra $cl(\Omega)$ and thus a two valued finitely additive measure on $A$. We say that $\phi$ is a two-valued finitely additive measure on $A$ if

1. $\phi : A \to \{0, 1\}$
2. when $a, b \in A$ and $a \land b = 0$, $\phi(a \lor b) = \phi(a) + \phi(b)$.
3. $\phi(1) = 1$.

We obtain the Stone space $\Omega$ by reversing the above idea. The product space $\{0, 1\}^A$ is compact (by the Tychonoff product theorem) and totally disconnected. Let

$$\Omega = \{\phi \in \{0, 1\}^A : \phi \text{ is a two-valued finitely additive measure}\}.$$

$\Omega$ is a closed subset of $\{0, 1\}^A$, (this is easily verified) and thus $\Omega$ is a compact totally disconnected space. If $a \in A$ we define

$$\overline{a} = \{\phi \in \Omega : \phi(a) = 1\}.$$

The map $a \to \overline{a}$ becomes our Boolean isomorphism from $A$ to $cl(\Omega)$. It is easily seen that each $\overline{a} \in cl(\Omega)$ and that the map is a Boolean homomorphism. There are two tricky points here. To show that the map is one-to-one we must show that for every $a \in A$ with $a \neq 0$, $\overline{a} \neq \phi$, i.e. there exists $\phi \in \Omega$, such that $\phi(a) = 1$. To do this use Zorn’s Lemma to show that $a \in U$, where $U$ is a maximal collection in $A$ with the finite intersection property $(a_1, \ldots, a_n \in U$ implies that $a_1 \land a_2 \land \cdots \land a_n \neq 0$). Such a collection $U$ is called an \textit{ultrafilter}. Then show that if

$$\phi(b) = \begin{cases} 1 & \text{if } b \in U \\ 0 & \text{if } b \notin U \end{cases}$$

then $\phi \in \Omega$. Since $a \in U, \phi(a) = 1$. The second tricky point is that of showing that

$$\overline{A} = \{\overline{a} : a \in A\} = cl(\Omega).$$

In any totally disconnected compact Hausdorff space $\Omega$ if an algebra $B$ of clopen sets separates points, then $B = cl(\Omega)$. After verifying this, note that $\overline{A}$ is an algebra of clopen sets and $\overline{A}$ separate points.
After our crash course in Stone spaces, let us look at what happens in the case when $\mathcal{A}$ is a Boolean $\sigma$-algebra. Letting $\Omega$ denote the Stone space, we let

$$
\mathcal{B} = \{ E \subseteq \Omega : \text{there exists } F \in \text{cl}(\Omega) \text{ such that } E \Delta F \text{ is nowhere dense} \}.
$$

The collection $\mathcal{B}$ is a $\sigma$-algebra in $\Omega$. This is easily verified using the fact that $\mathcal{A}$ is a Boolean $\sigma$-algebra. When $F \in \mathcal{B}$, there is only one $E \in \text{cl}(\Omega)$ so that $F \Delta E$ is nowhere dense. We denote $E$ by $[F]$. If $\mathcal{A}$ is a submeasure algebra, then there is a strictly positive continuous submeasure $\nu$ on $\mathcal{A}$. Regarding $\nu$ as defined on the algebra $\text{cl}(\Omega)$ we define $\bar{\nu}$ on $\mathcal{B}$ by

$$
\bar{\nu}(F) = \nu([F]).
$$

It is easily seen that $\bar{\nu}$ is a continuous submeasure on $\mathcal{B}$ and $\mathcal{A}$ is Boolean isomorphic to the submeasure algebra induced by $(\bar{\nu}, \mathcal{B}, \Omega)$.

Definition 2.2.

A Boolean $\sigma$-algebra $\mathcal{A}$ is called a measure algebra if there exists $\mu$ a finitely additive measure on $\mathcal{A}$ such that $\mu$ is strictly positive and continuous.

A measure algebra is clearly a submeasure algebra. By the preceding remarks, the converse of this is equivalent to Maharam's Problem.

Maharam's Problem II:
Is every submeasure algebra a measure algebra?

In order to fully savor this version of Maharam's Problem, let us further investigate some of the properties of submeasure algebras.

Definition 2.3.

Suppose $\mathcal{A}$ is a Boolean algebra. If $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{C}$ is called a (pairwise) disjoint collection if $a, b \in \mathcal{A}$ implies $a \land b = 0$. A disjoint collection $\mathcal{C}$ is called a partition if $\forall C \in \mathcal{C}$ is a countable, $\mathcal{A}$ is said to have the countable chain condition (c.c.c).

Definition 2.4.

Suppose $\nu$ is a submeasure on a Boolean algebra $\mathcal{A}$. $\nu$ is exhaustive if whenever $< a_n >$ is a pairwise disjoint sequence in $\mathcal{A}$,

$$
\lim_{n \to \infty} \nu(a_n) = 0.
$$

Proposition 2.5.

(1) If $\nu$ is a continuous submeasure on a Boolean $\sigma$-algebra $\mathcal{A}$, then $\nu$ is exhaustive

(2) Every submeasure algebra satisfies c.c.c.
Proof:

(1) Let \( \langle a_n \rangle \) be a pairwise disjoint sequence in \( A \) and suppose \( \nu(a_n) \to 0 \). By passing to a subsequence if necessary we may suppose that \( \nu(a_n) \geq \epsilon \) for all \( n \) where \( \epsilon > 0 \). Let \( b_n = \bigvee \{ a_k : k \leq n \} \). Then \( \nu(b_n) \geq \epsilon, b_1 \geq b_2 \geq \cdots \) and \( \land \{ b : n \in N \} = 0 \). We have a contradiction since \( \nu \) is continuous.

(2) Suppose that \( A \) is a submeasure algebra with strictly positive and continuous submeasure \( \nu \). Let \( C \) be a disjoint collection in \( A \). For each \( \epsilon > 0 \),

\[ \{ a \in C : \nu(a) \geq \epsilon \} \text{ is finite.} \]

Otherwise we can select a pairwise disjoint sequence \( \langle a_n \rangle \) from \( C \) so that \( \nu(a_n) \geq \epsilon \) for all \( n \). But then \( C \) is countable.

Proposition 2.6.

Every Boolean \( \sigma \)-algebra \( A \) with c.c.c. is complete. Also if \( C \subseteq A \), there exists a sequence \( \langle a_n \rangle \) in \( C \) such that \( \forall C = \forall \{ a_n : n \in N \} \). Thus submeasure algebras are complete and have the above property.

Proof:

Let \( C \subseteq A \). By Zorn's Lemma we may select a maximal pairwise disjoint collection \( D \) with the property that whenever \( d \in D \), there exists \( c \in C \) such that \( d \leq c \). \( D \) is countable so \( D \) has a least upper bound \( a \). Suppose there is some \( c \in C \) such that \( c \leq a \) fails. Then \( c \land a' \neq 0 \) and we may enlarge \( D \) to \( D \cup \{ c \land a' \} \). This would contradict the maximality of \( D \). Hence \( C \subseteq a \). Now \( D = \{ d_n : n \in N \} \) and each \( d_n \leq c_n \) for some \( c_n \in C \). Also

\[ a = \bigvee \{ d_n : n \in N \} \leq \bigvee \{ c_n : n \in N \} \leq a. \]

Thus

\[ a = \bigvee \{ c_n : n \in N \} = \forall C. \]

Definition 2.7.

Suppose \( A \) is a Boolean \( \sigma \)-algebra with c.c.c. If \( \pi \) is a partition in \( A \) and \( a \in A \), we say that \( a \) is finitely covered by \( \pi \) if there exists a finite set \( \{ a_1, \ldots, a_n \} \subseteq \pi \) such that

\[ a \subseteq \bigvee \{ a_i : 1 \leq i \leq n \}. \]

We say that \( A \) satisfies the generalized distributive law (g.d.l.) if whenever \( \langle \pi_n \rangle \) is a sequence of partitions, there is a single partition \( \pi \) such that if \( a \in \pi \), \( a \) is finitely covered by each \( \pi_n \).

Notes. (1) The partitions mentioned above are all countable. (2) The above definition of the g.d.l. is not the standard definition. But in Boolean \( \sigma \)-algebras with c.c.c. it is equivalent to the standard definition.
Proposition 2.8.

(1) If $\mathcal{A}$ is a Boolean $\sigma$-algebra and $\nu$ is a continuous submeasure on $\mathcal{A}$, then for every sequence $<a_n>$ in $\mathcal{A}$ with $a = \vee\{a_n : n \in N\}$,
\[
\nu(a) \leq \sum_{n=1}^{\infty} \nu(a_n)
\]

i.e. $\nu$ is countably subadditive.

(2) If $\mathcal{A}$ is a submeasure algebra, then $\mathcal{A}$ satisfies the g.d.l.

Proof:

(1) Let $b_n = a \land a'_1 \land \cdots \land a'_n$. Then $b_1 \geq b_2 \geq \cdots$ and $\land\{b_n : n \in N\} = 0$. Hence
\[
\lim_{n \to \infty} \nu(b_n) = 0. \text{ Now}
\]
\[
\nu(a) = \nu(a_1 \lor \cdots \lor a_n \lor b_n) \leq \sum_{i=1}^{n} \nu(a_i) + \nu(b_n).
\]
Thus $\nu(a) \leq \sum_{n=1}^{\infty} \nu(a_n)$.

(2) Let $\pi = \{a_n : n \in N\}$ be a partition.

If $b_n = a_1 \lor \cdots \lor a_n$, then $\lim_{n \to \infty} \nu(b_n) = 0$.

($\nu$ is our strictly positive continuous submeasure on $\mathcal{A}$). Since each $b_n$ is finitely covered by $\pi$, we may for any $\varepsilon > 0$ select $a \in \mathcal{A}(a = b_n$ for suitable $n$) such that $\nu(a') < \varepsilon$ with $a$ finitely covered by $\pi$. Now if $\varepsilon > 0$ and $<\pi_n>$ is a sequence of partitions we may, for each $n$, select $a_n$ such that $a_n$ is finitely covered by $\pi_n$ and $\nu(a'_n) < \frac{\varepsilon}{2^n}$.

Let $a = \land\{a_n : n \in N\}$. $a$ is finitely covered by each $\pi_n$. Also,
\[
\nu(a') = \nu(\lor\{a'_n : n \in N\}) \leq \sum_{n=1}^{\infty} \nu(a'_n) < \varepsilon.
\]
Thus one can select a sequence $<a_n>$ in $\mathcal{A}$ such that each $a_n$ is finitely covered by all the partitions and $\lim_{n \to \infty} \nu(a'_n) = 0$. At last let $\pi = \{a_n \land a'_1 \land \cdots \land a'_{n-1} : n \in N\}$.

In 1947, D. Stone (then D. Maharam) posed her famous problem in [7]. In 1937 J. von Neumann posed another famous unsolved problem in the Scottish Book (Problem 163).

**Von Neumann's Problem:** If $\mathcal{A}$ is a Boolean $\sigma$-algebra which satisfies the c.c.c. and g.d.l., is $\mathcal{A}$ a measure algebra?

A positive answer to von Neumann's Problem would also provide a positive answer to Maharam's Problem (since submeasure algebras satisfy the c.c.c. and the g.d.l.). Before providing the current status of this attempt at a rather lush characterization of measure algebras, we present yet another clue in the puzzle.

Proposition 2.9.

If $\mathcal{A}$ is a Boolean $\sigma$-algebra with the g.d.l. and $\mathcal{A}$ admits a strictly positive finitely additive measure then $\mathcal{A}$ is a measure algebra.
Proof:

Note that the c.c.c. follows automatically from the existence of a strictly positive finitely additive measure. Let \( \phi \) denote this finitely additive measure. For each \( a \in A \) define

\[
\phi^*(a) = \inf_{\pi} \sum_{a \in \pi} \phi(a)
\]

where the infimum is taken over all partitions \( \pi \) of \( a \) (\( \pi \) is a partition of \( a \) if \( \pi \cup \{a'\} \) is a partition, i.e. \( \pi = \{a_n : n \in \mathbb{N} \} \) a disjoint collection such that \( \forall \pi = a \)). It is easily verified that \( \phi^* \) is strictly positive. To see that, let \( a \in A \sim \{0\} \) and suppose partitions \( \pi_n \) of \( a \) are chosen so that

\[
\phi^*(a) = \lim_{n \to \infty} \sum_{b \in \pi_n} \phi(b).
\]

Since \( A \) satisfies the g.d.l. there exists \( c \in A \) such that \( c \neq 0, c \leq a \) and \( c \) is finitely covered by each \( \pi_n \). Then

\[
\phi(c) \leq \sum_{b \in \pi_n} \phi(b)
\]

and thus \( 0 < \phi(c) \leq \phi^*(a) \). It now follows that \( A \) is a measure algebra.

Question. If \( A \) is a Boolean algebra with the c.c.c., does \( A \) admit a strictly positive finitely additive measure?

A positive answer to this question would resolve all of the problems positively. Counting the pages left in this article along with a little gamesmanship tells the astute reader that the answer must be negative! It is. The Souslin Hypothesis is independent of the axioms of set theory. In her 1947 paper [7], D. Stone produced an example of a Boolean \( \sigma \)-algebra \( A \) with the c.c.c. and the g.d.l. such that \( A \) is not even a submeasure algebra. But the algebra was constructed assuming the negation of the Souslin Hypothesis. Thus there is no hope of a positive solution to the question nor to a positive solution to von Neumann’s Problem. In [4] Gaifman constructed a Boolean algebra \( A \) with the c.c.c. and such that \( A \) does not admit a strictly positive finitely additive measure.

At this point it might seem that the waters have become a bit murky. But, in fact, it is precisely at this point that the game is afoot. The true motives of the Maharam Problem enthusiast are all too apparent. Measure algebras are fundamental objects in mathematics. Boolean algebras satisfying von Neumann’s conditions are very nearly measure algebras (D. Stone’s example is the only example known other than measure algebras). Axioms independent of the axioms of set theory (such as the negation of the Souslin Hypothesis) are not user friendly. A negative answer to Maharam’s Problem would provide a highly structured Boolean algebra of considerable interest and of fundamental importance. The next clue to the puzzle comes from J. L. Kelley.

3. KELLEY’S CHARACTERIZATION OF MEASURE ALGEBRA.

It is clear that an algebraic characterization of measure algebras would be very useful at this point. The search for such conditions has been a major theme so far. The von
Neumann conditions unfortunately do not suffice. In her 1947 paper [7], D. Stone gave an algebraic characterization of measure algebras. However, the conditions obtained are rather complicated. In 1959, J. L. Kelley gave an algebraic characterization of measure algebras which is the best obtained to this date (and perhaps the best possible). The idea is to obtain a condition on a Boolean algebra equivalent to its having a strictly positive finitely additive measure. To motivate this we consider an algebra $\mathcal{A}$ of subsets of a set $\Omega$ (every Boolean algebra is isomorphic to an algebra of sets). Suppose $\phi$ is a finitely additive probability measure on $\mathcal{A}(\phi(\Omega) = 1)$. Also suppose $\mathcal{C}$ is a collection of subsets in $\mathcal{A}$ such that $\phi(E) \geq \lambda$ for every $E \in \mathcal{C}$. Then if $(E_1, \ldots, E_n)$ is a finite sequence of members of $\mathcal{C}$, then

$$\int \sum_{j=1}^{n} 1_{E_j} d\phi = \sum_{i=1}^{n} \phi(E_i) \geq n\lambda.$$

Thus

$$\max_{x \in \Omega} \sum_{i=1}^{n} 1_{E_i}(x) \geq n\lambda,$$

so that there exists $x \in \Omega$ such that

$$|\{i : x \in E_i\}| \geq n\lambda.$$

Equivalently for some $K \geq \lambda n$ there is a finite subsequence $(E_{i_1}, \ldots, E_{i_K})$ such that

$$E_{i_1} \cap \cdots \cap E_{i_K} \neq \emptyset.$$

**Definition 3.1.**

Suppose $\mathcal{A}$ is a Boolean algebra and $\mathcal{C} \subset \mathcal{A}$. We say that $\mathcal{C}$ has intersection number $\iota(i, (\mathcal{C})) = \lambda$ for $\lambda \geq 0$ if $\lambda$ is the largest nonnegative number so that whenever $(a_1, \ldots, a_n)$ is a finite sequence in $\mathcal{C}$, there is a finite subsequence $i_1 < \cdots < i_K$ with $K \geq \lambda n$ such that

$$a_{i_1} \land \cdots \land a_{i_K} \neq 0.$$

**Notes.**

(1) In a finite sequence $(a_1, \ldots, a_n)$ a member of $\mathcal{C}$ can appear more than once, i.e. the $a_i$'s need not be distinct.

(2) $\iota(\mathcal{C})$ is a supremum of nonnegative numbers. It is easily seen that the condition holds for this supremum.

The following is a standard result which will prove very useful throughout the paper.

**Proposition 3.2.**

Suppose $\Omega$ is a compact Hausdorff space. Also suppose that $\mathcal{C} \subset C(\Omega)$ such that whenever $(f_1, \ldots, f_n)$ is a finite sequence in $C(\Omega)$, then

$$\sum_{i=1}^{n} f_i(x) \geq 0$$
for some $x \in \Omega$, i.e.

$$\max \sum_{i=1}^{n} f_i \geq 0.$$  

Then there exists a regular Borel probability measure $\mu$ on $\Omega$ such that for every $f \in \mathcal{C}$

$$\int f \, d\mu \geq 0.$$  

Proof:

We first claim that if $f$ is in the convex hull of $\mathcal{C}, \text{co}\mathcal{C}$, then $\max f \geq 0$. To see this suppose

$$f = \sum_{i=1}^{n} \alpha_i f_i$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. Without loss of generality (by an easy limiting argument) we may assume each $\alpha_i$ is rational. Using a common denominator we may suppose $\alpha_i = \frac{n_i}{N}$ where

$$\sum_{i=1}^{m} n_i = N.$$  

If we consider the finite sequence $\langle g_1, \cdots, g_N \rangle$ where each $f_i$ is repeated $n_i$-times

$$\sum_{i=1}^{n} \alpha_i f_i = \frac{1}{N} \sum_{j=1}^{N} g_j.$$  

Thus,

$$\max \sum_{i=1}^{n} \alpha_i f_i \geq 0.$$  

Let $F = \{1 + \alpha f + g : \alpha \geq 0, f \in \text{co}\mathcal{C}, g \in C(\Omega) \text{ and } g \geq 0\}$. Let $B$ denote the open unit ball of $C(\Omega)$ with the supremum norm. Since $F$ and $B$ are convex, $B$ is open and $B \cap F = \phi$, there exists a continuous linear functional $\lambda$ with norm 1 so that

$$\sup \lambda(B) = 1 \leq \inf \lambda(F)$$

by the Hahn-Banach Theorem. $1 \in F$ so $\lambda(1) \geq 1$. Since $\| \lambda \| \leq 1 \lambda(1) \leq 1$. So $\lambda(1) = 1$. If $g \in C(\Omega)$ and $g \geq 0$, then

$$\lambda(g) + 1 = \lambda(g + 1) \geq 1$$

since $1 + g \in F$. Thus $\lambda(g) \geq 0$. By the Riesz Representation Theorem, there is a regular Borel probability measure $\mu$ on $\Omega$ such that

$$\lambda(f) = \int f \, d\mu.$$
for each $f \in C(\Omega)$. Now if $f \in \mathcal{C}, 1 + f \in F$. Thus
\[ \lambda(f) + 1 = \lambda(f + 1) \geq 1 \]
so $\lambda(f) \geq 0$, i.e.
\[ \int f \, d\mu \geq 0. \]

**Proposition 3.3.**

Suppose $\mathcal{A}$ is a Boolean algebra and $\mathcal{C} \subset \mathcal{A}$. For $\lambda \geq 0$, $i(\mathcal{C}) = \lambda$ if and only if there exists a finitely additive probability measure $\phi$ on $\mathcal{A}$ such that $\phi(a) \geq \lambda$ for all $a \in \mathcal{C}$.

**Proof:**

Suppose $i(\mathcal{C}) = \lambda$. Without loss of generality we may assume that $\mathcal{A}$ is the algebra $cl(\Omega)$ where $\Omega$ is a compact Hausdorff space. For each clopen set $E \in \mathcal{C}, 1_E \in C(\Omega)$. Let $\mathcal{C}' = \{1_E - \lambda 1 : E \in \mathcal{C}\}$. If $(E_1, \cdots, E_m)$ is a finite sequence in $\mathcal{C}$ then since $i(\mathcal{C}) = \lambda$
\[ \max \sum_{i=1}^m 1_{E_i} \geq m \lambda, \]
i.e.
\[ \max \sum_{i=1}^m (1_{E_i} - \lambda 1) \geq 0. \]

Applying Proposition 3.2 to $\mathcal{C}'$ there exists a regular Borel measure $\mu$ on $\Omega$ such that
\[ \int 1_E - \lambda 1 \, d\mu \geq 0 \]
for each $E \in \mathcal{C}$, i.e. $\mu(E) \geq \lambda$ for each $E \in \mathcal{C}$.

Our opening remarks that motivated the idea of intersection number constitute a proof of the converse.

**Theorem 3.4.**

(J. L. Kelley): A Boolean algebra $\mathcal{A}$ is a measure algebra if and only if $\mathcal{A}$ is a Boolean $\sigma$-algebra such that:

1. $\mathcal{A}$ satisfies the g.d.l. and
2. $\mathcal{A} \sim \{0\} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ where $i(\mathcal{C}_n) > 0$ for each $n$.

**Proof:**

If $\mathcal{A}$ is a Boolean algebra satisfying (2), then for each $N$ there exists a finitely additive probability measure $\phi_n$ such that $\phi_n(a) > 0$ for each $a \in \mathcal{C}_n$. Now let $\phi = \sum_{n=1}^{\infty} 2^{-n} \phi_n$. $\phi$ is a strictly positive finitely additive probability measure on $\mathcal{A}$. Conversely, if $\phi$ is a strictly positive finite measure on $\mathcal{A}$, we may take $\mathcal{C}_n = \{a \in \mathcal{A} : \phi(a) \geq \frac{1}{n}\}$.  

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Notes

Condition (2) characterizes Boolean algebras which possess a strictly positive finitely additive probability measure.

Kelley’s Theorem can be applied directly to submeasure algebras. Suppose every submeasure algebra is a measure algebra. Let \( \nu \) be a strictly positive continuous submeasure on a Boolean \( \sigma \)-algebra \( \mathcal{A} \) and let \( \mu \) be an equivalent probability measure. For every \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that if \( \mu(a) < \delta \), then \( \nu(a) < \varepsilon \). If we let

\[
\mathcal{C} = \{ a \in \mathcal{A} : \nu(a) \geq \varepsilon \},
\]

then \( a \in \mathcal{C} \) implies \( \mu(a) \geq \delta \). Thus \( i(\mathcal{C}) \geq \delta \). Thus we obtain another formulation of Maharam’s Problem.

**Maharam’s Problem III:**
If \( \nu \) is a continuous strictly positive submeasure on a Boolean \( \sigma \)-algebra \( \mathcal{A} \) and \( \varepsilon > 0 \) is

\[
i(\{ a \in \mathcal{A} : \nu(a) \geq \varepsilon \}) > 0?
\]

4. **EXHAUSTIVE SUBMEASURES**

Recall that every continuous submeasure is exhaustive. Suppose that \( \mathcal{A} \) is a Boolean algebra and \( \nu \) is an exhaustive submeasure on \( \mathcal{A} \). If \( \nu \) is not strictly positive we could define an equivalence relation on \( \mathcal{A} \) by saying

\[
a \equiv b \text{ if } \nu(a \Delta b) = 0.
\]

\( \nu \) then becomes a positive exhaustive submeasure on the Boolean algebra of equivalence classes. So we shall assume that \( \nu \) is strictly positive at the outset. If we define

\[
d(a, b) = \nu(a \Delta b)
\]

for \( a, b \in \mathcal{A} \), then \( (\mathcal{A}, d) \) is a metric space. The metric completion \( \overline{\mathcal{A}} \) is easily verified to be a Boolean algebra (the operations are obtained by taking limits). Also, it is easily shown that \( \nu \) extends to a strictly positive submeasure (still called \( \nu \)) on \( \overline{\mathcal{A}} \). Also \( \nu \) is still exhaustive on \( \overline{\mathcal{A}} \) and

\[
d(a, b) = \nu(a \Delta b) \text{ for all } a, b \in \mathcal{A}.
\]

**Proposition 4.1.**

Suppose that \( \nu \) is a strictly positive exhaustive submeasure on a Boolean algebra \( \mathcal{A} \) and with

\[
d(a, b) = \nu(a \Delta b),
\]

\( \mathcal{A} \) is complete with respect to \( d \). Then \( \mathcal{A} \) is a submeasure algebra and \( \nu \) is a continuous submeasure on \( \mathcal{A} \).
Proof:

Suppose that \(< a_n >\) is a decreasing sequence in \(A\). It will suffice to prove that \(\{a_n : n \in N\}\) has a greatest lower bound. If we show that \(< a_n >\) is Cauchy, the limit is easily seen to be the greatest lower bound. Suppose \(< a_n >\) is not Cauchy. Then there exists \(\epsilon > 0\) such that for every \(n\)

\[
\lim_{k \to \infty} \nu(a_n \wedge a'_{K}) > \epsilon.
\]

Thus we can select a subsequence \(< a_{n_k} >\) so that for each \(k\)

\[
\nu(a_{n_k} \wedge a'_{n_k+1}) > \epsilon.
\]

But then if \(b_k = a_{n_k} \wedge a'_{n_k+1}\), the sequence \(< b_k >\) is pairwise disjoint and \(\nu(b_k) > \epsilon\) for each \(k\). Thus \(A\) is a Boolean \(\sigma\)-algebra. Now let us suppose that

\[
\wedge\{a_n : n \in N\} = 0,
\]

with \(< a_n >\) still decreasing. But as we observed above, 0 is the metric limit of the sequence \(< a_n >\), i.e.

\[
\lim_{n \to \infty} \nu(a_n) = \lim_{n \to \infty} d(a_n, 0) = 0.
\]

Thus \(\nu\) is continuous.

We now have a blueprint for constructing submeasure algebras. Construct an exhaustive submeasure on an algebra and take the metric completion. But how do we guarantee that the completion \(A\) does not have an equivalent measure? The answer is that in the original algebra \(A\) we have \(\epsilon > 0\) so that

\[
i(\{a : \nu(a) \geq \epsilon\}) = 0.
\]

Definition 4.2.

A submeasure \(\nu\) on a Boolean algebra \(A\) is pathological if for some \(\epsilon > 0\)

\[
i(\{a \in A : \nu(a) \geq \epsilon\}) = 0.
\]

Note that if (without any loss of generality) \(A\) is an algebra of sets and \(\nu\) is pathological on \(A\), then there is a fixed \(\epsilon > 0\) so that if \(\lim_{n \to \infty} \delta_n = 0\) with \(\delta_n > 0\) there exists a finite sequence \((A_{i_1}, \ldots, A_{i_k}, n)\) so that

\[
\sum_{i=1}^{K_n} 1_{A_{i_n}} \leq \delta_n K_n
\]

but \(\nu(A_{i_n}) \geq \epsilon\). So if \(\nu\) is a pathological on \(A\), it is pathological on the countable algebra generated by the sets \(\{A_{i_n}\}\), i.e. \(\nu\) is pathological on a countable subalgebra. Piecing all of this together we can obtain yet another version of Maharam's Problem.
Maharam’s Problem IV:

Does every pathological submeasure on a countable algebra of sets fail to be exhaustive?

Examples of pathological submeasures have been obtained by Christensen and Herer [1], Popov [8], and Talagrand [11]. Unfortunately, none of these are exhaustive.

5. THE CONTROL MEASURE PROBLEM.

Let $X$ be a vector space over the real numbers and suppose $\| \cdot \|$ is a nonnegative real valued function on $X$. Suppose also that

1. $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in X$
2. $\| \alpha x \| \leq \| x \|$ when $x \in X$ and $\alpha$ real with $| \alpha | \leq 1$
3. $\| x \| = 0$ implies $x = 0$ and
4. if $x \in X$ and $\langle \alpha_n \rangle$ is a sequence of scalars such that $\lim_{n \to \infty} \alpha_n = 0$ then

$$\lim_{n \to \infty} \| \alpha_n x \| = 0$$

Then $\| \cdot \|$ is called an $F$-norm on $X$. The metric topology defined by $d(x, y) = \| x - y \|$ provides $X$ with a linear topology. Suppose $\mathcal{A}$ is an algebra of subsets of a set $\Omega$ and

$$\mu : \mathcal{A} \to X$$

such that for every disjoint pair of sets $A, B \in \mathcal{A}$

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Then $\mu$ is a finitely additive $X$-valued vector measure. $\mu$ is said to be exhaustive if whenever $\langle A_n \rangle$ is a pairwise disjoint collection in $\mathcal{A}$

$$\lim_{n \to \infty} \mu(A_n) = 0.$$

A finitely additive measure $\phi$ on $\mathcal{A}$ is called a control measure for $\mu$ if $\lim_{n \to \infty} \phi(A_n) = 0$ implies $\lim_{n \to \infty} \| \mu(A_n) \| = 0$ for every sequence $\langle A_n \rangle$ in $\mathcal{A}$. We define a submeasure $\nu$ on $\mathcal{A}$ by

$$\nu(A) = \sup\{ \| \mu(B) \| : B \subset A, B \in \mathcal{A} \}.$$

Note that $\nu$ is exhaustive if $\mu$ is exhaustive. If Maharam’s Problem has a positive solution and $\mu$ is exhaustive, then $\nu$ has an equivalent finitely additive measure $\phi$, i.e., a control measure for $\mu$. Suppose there is a counterexample to Maharam’s Problem, let us say a continuous submeasure $\nu$ on a $\sigma$-algebra of sets $\mathcal{A}$ in a set $\Omega$. Then the space of measurable functions (actually equivalence classes of functions) $L_0(\nu)$ can be provided with an $F$-norm so that

$$\lim_{n \to \infty} f_n = 0 \text{ if and only if }$$

for every $\epsilon > 0$

$$\lim_{n \to \infty} \nu\{ x : | f_n(x) | \geq \epsilon \} = 0.$$

The vector measure $\mu : \mathcal{A} \to L_0(\nu)$ defined by

$$\mu(A) = 1_A$$

is exhaustive. The measure $\mu$ would have no control measure.
Maharam's Problem V: (The Control Measure Problem)

Does every exhaustive finitely additive vector measure have a control measure?

There are other equivalent versions of the control measure problem. We have presented just one of these. The Control Measure Problem provides a powerful motive for wanting a positive solution to Maharam's Problem. As we shall see in the next section, it is possible that the problem could have a pleasing solution that makes everyone happy.

6. UNIFORMLY EXHAUSTIVE SUBMEASURES.

If \( \nu \) is a submeasure on a Boolean algebra \( \mathcal{A} \), we say that \( \nu \) is uniformly exhaustive for every \( \epsilon > 0 \) there exists a positive integer \( N \) such that whenever \( \{a_1, \ldots, a_N\} \) is a pairwise disjoint collection in \( \mathcal{A} \), there exists some \( a_i \) with \( \nu(a_i) < \epsilon \). Of course, uniformly exhaustive submeasures are exhaustive. It is easy to see that \( \nu \) is uniformly exhaustive if and only if there exists a decreasing sequence \( \epsilon_n \) of positive numbers so that

\[
\lim_{n \to \infty} \epsilon_n = 0 \quad \text{and whenever} \quad \{a_n\} \quad \text{is a pairwise disjoint sequence in} \quad \mathcal{A} \quad \text{with the sequence} \\
\nu(a_n) \quad \text{decreasing, then} \quad \nu(a_n) \leq \epsilon_n \quad \text{for each} \quad n. 
\]

For instance, if \( \nu \) is a finitely additive probability measure, we can take \( \epsilon_n = \frac{1}{n} \). It is also easy to see that if a submeasure \( \nu \) has an equivalent finitely additive measure, then \( \nu \) is uniformly exhaustive.

The notion of uniformly exhaustive submeasures was coined by M. Talagrand. He also posed the natural question: does every uniformly exhaustive submeasure have an equivalent finitely additive measure? The answer is yes and this result appears in [5].

Theorem 6.1. A submeasure is uniformly exhaustive if and only if it has an equivalent finitely additive measure.

The entire problem (and its solution) hinges on the existence of a pathological combinatorial object known as a concentrator. It is ironic that to deny the existence a uniformly exhaustive submeasure that is pathological, one uses combinatorial pathology. We also note that Kelley's theorem plays a major role in the proof. N. J. Kalton used the existence of concentrators to resolve two other open problems in F-space theory. He showed that every countably additive vector measure with compact range has a control measure and he showed that \( c_0 \) is a K-space. These results also appear in [5]. It is pleasant to see the one notion (of concentrator) cracks so many nuts! In the last section we shall see that the combinatorial aspect of Maharam's Problem continues to play a major role.

At last, let us notice that Theorem 6.1 gives us a much simpler characterization of pathological submeasures. Namely, if \( \nu \) is a pathological submeasure on a Boolean algebra \( \mathcal{A} \) then there exists some \( \epsilon > 0 \) so that for every positive integer \( N \) there exists \( \{a_1, \ldots, a_N\} \) a pairwise disjoint collection in \( \mathcal{A} \) so that \( \nu(a_i) > \epsilon \) for each \( i \). In other words, there exist arbitrarily large partitions in \( \mathcal{A} \) with each piece of one of these having value at least \( \epsilon \). It is enticing to believe that perhaps one can jump from arbitrarily large finite partitions with this property to a single infinite partition. This, of course, is now Maharam's Problem. D. H. Fremlin believed that this was the real problem long before Theorem 6.1 was proved [see 2].

In practice, exhaustive vector measures tend to give rise to uniformly exhaustive submeasures. So Theorem 6.1 suffices for the purposes of a vector measure theorist. If a
counterexample to Maharam's Problem exists, then the hoped-for submeasure algebra (that is not a measure algebra) would make measure theorists very happy. The sentimental favorite of all possible solutions is a counterexample. We shall not prove Theorem 6.1 here. Instead, we refer the reader to [5].

7. PATHOLOGICAL SUBMEASURES ON THE CANTOR SET.

We begin by demonstrating that if there is a pathological exhaustive submeasure, then there is a pathological exhaustive submeasure on the clopen subsets of the Cantor set. Recall that any compact metric space that is totally disconnected and that has no isolated points is homeomorphic to the Cantor set. Suppose $\mathcal{A}$ is an algebra of subsets of a set $\Omega$ and suppose that $\nu$ is a pathological exhaustive submeasure on $\mathcal{A}$. Since $\nu$ is not uniformly exhaustive there exists $\varepsilon > 0$ and a sequence $\langle \pi_n \rangle$ of finite partitions in $\mathcal{A}$ such that

1. for every $A \in \pi_n$, $\nu(A) \geq \varepsilon$
2. if $|\pi_n| = K_n$, then $\lim_{n \to \infty} K_n = \infty$.

We let $[K]$ denote the set $\{1, 2, \ldots, K\}$. Also let $\pi_n = \{A_{1n}, A_{2n}, \ldots, A_{Kn}\}$. Our copy of the Cantor set will be $K = \prod_{n=1}^{\infty} [K_n]$. Notice that if $\mathcal{A}_0$ is the algebra of sets generated by the collection of sets $\cup\{\pi_n : n \in \mathbb{N}\}$ then $\nu$ is an exhaustive pathological submeasure on $\mathcal{A}_0$. Define a map

$$\phi : \Omega \to \prod_{n=1}^{\infty} [K_n] = K$$

as follows. For $1 \leq i \leq K_n$, let

$$S_{in} = \{x : x \in K : x(n) = i\}.$$ 

Each $S_{in}$ is a coordinate set in $K$.

If $x \in \Omega$, let $\phi(x)$ be the single point in the set

$$\cap\{S_{in} : x \in A_{in}, 1 \leq i \leq K_n, n \in \mathbb{N}\}$$

i.e. $\phi(x)(n) = i$ if $x \in A_{in}$ for each $n$. Now each $S_{in}$ is a clopen set in $K$. We denote the collection consisting of the coordinate sets $S_{in}$ by $\mathcal{C}$. Since the sets $\mathcal{C}$ separate points in $K$, the algebra of sets generated by $\mathcal{C}$ is $cl(K)$. Also for each $S_{in} \in \mathcal{C}$, $\phi^{-1}(S_{in}) = A_{in}$. Thus if $E \in cl(K)$, then $\phi^{-1}(E) \in \mathcal{A}_0$. We now define $\nu_1$ on $cl(K)$ by

$$\nu_1(E) = \nu(\phi^{-1}(E))$$

for $E \in cl(K)$. Clearly $\nu_1$ is an exhaustive submeasure on $cl(K)$. Also for each $n$, the collection

$$\mathcal{C}_n = \{S_{1n}, \ldots, S_{Kn}\}$$

is a partition of $K$ such that $\nu_1(S_{in}) \geq \varepsilon, 1 \leq i \leq K_n$. So $\nu_1$ is not uniformly exhaustive. Thus $\nu_1$ is pathological.

**Maharam's Problem VI:**
There does not exist an exhaustive submeasure $\nu$ on $cl(K)$ and $\varepsilon > 0$ such that for every coordinate set $S \in \mathcal{C},$

$$\nu(S) \geq \varepsilon$$
Note.

If we wish to have \( \nu(S) = 1 = \nu(K) \), we could replace \( \nu \) by \( \nu_0 \) where

\[
\nu_0(E) = \max \{ \epsilon^{-1}\nu(E), 1 \}.
\]

**Proposition 7.1.**

If \( \nu \) is an exhaustive submeasure on \( cl(K) \), then there is a unique extension of \( \nu \) to a continuous submeasure (also denoted by \( \nu \)) on the Borel sets of \( K \) such that \( \nu \) is regular, i.e. for each Borel set \( A \)

\[
\nu(A) = \sup\{\nu(F) : F \subset A, \text{ } F \text{ compact} \} = \inf\{\nu(O) : A \subset O, \text{ } O \text{ open} \}
\]

**Proof:**

If \( O \) is open, define

\[
\nu(O) = \sup\{\nu(E) : E \subset O, E \in cl(K) \}
\]

and if \( F \) is closed define

\[
\nu(F) = \inf\{\nu(E) : F \subset E, E \in cl(K) \}.
\]

Notice that these extensions of \( \nu \) agree on the sets that are both closed and open, i.e. the clopen sets. We say that a set \( A \) is \( \nu \)-measurable if for every \( \epsilon > 0 \), there exists sets \( F \) closed and \( O \) open such that \( F \subset A \subset O \) and

\[
\nu(O - F) < \epsilon.
\]

Note that if \( O \) is open and \( \epsilon > 0 \), there exists \( E \in cl(K) \) such that \( E \subset O \) and

\[
\nu(O - E) < \epsilon.
\]

If not, we may then inductively select a pairwise disjoint sequence \(< E_n >\) of clopen subsets of \( O \) such that \( \nu(E_n) > \epsilon \) for each \( n \). To see this, suppose \( E_1, \cdots, E_n \) have been selected. Then

\[
\nu(O - \bigcup_{i=1}^{n} E_i) > \epsilon.
\]

Hence there exists a clopen subset \( E_{n+1} \) of the open set \( O - \bigcup_{i=1}^{n} E_i \) so that \( \nu(E_{n+1}) \epsilon \). The selection of such a sequence in \( cl(K) \) is impossible since \( \nu \) is exhaustive.

It is easily seen that the \( \nu \)-measurable sets form an algebra. To show that the \( \nu \)-measurable sets form a \( \sigma \)-algebra (and thus the Borel sets are \( \nu \)-measurable) suppose that \(< A_n >\) is an increasing sequence of \( \nu \)-measurable sets. We let \( \epsilon > 0 \). Now for each \( n \), select \( F_n \) closed and \( O_n \) open such that \( F_n \subset A_n \subset O_n \) and

\[
\nu(O_n - F_n) < \epsilon \cdot 2^{-(n+1)}.
\]
Let $O = \bigcup_{i=1}^{\infty} O_i$. There exists $E \in \text{cl}(K)$ such that $E \subset O$ and $\nu(O - E) < \frac{\varepsilon}{2}$. Also, for some $n$,

$$E \subset \bigcup_{i=1}^{n} O_i.$$

Now $\bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{\infty} A_i \subset O$ and

$$\nu(O - \bigcup_{i=1}^{n} F_i) \leq \nu(O - E) + \sum_{i=1}^{n} \nu(O_i - F_i) < \varepsilon.$$

The above inequality follows from the subadditivity of $\nu$ on open sets (which is easily verified).

Now suppose that $\langle A_n \rangle$ is a decreasing sequence of Borel sets such that $\bigcap_{i=1}^{\infty} A_n = \emptyset$. The sequence $\langle \nu(A_n) \rangle$ is decreasing. Suppose

$$\lim_{n \to \infty} \nu(A_n) = \varepsilon$$

for some $\varepsilon > 0$. By our criteria for measurability there exists a sequence $\langle F_n \rangle$ of closed sets such that $F_n \subset A_n$ and

$$\nu(A_n - F_n) < \varepsilon 2^{-n+1}.$$

For any $n$,

$$\left( \bigcap_{i=1}^{n} F_i \right) \cup \bigcup_{k=1}^{n} (A_k - F_k) \supset A_n.$$

Thus $\nu(\bigcap_{i=1}^{n} F_i) > \nu(A_n) - \sum_{k=1}^{n} \nu(A_k - F_k) > \frac{\varepsilon}{2}$. Thus $\bigcap_{i=1}^{n} F_i \neq \emptyset$. But then $\bigcap_{i=1}^{n} A_n \neq \emptyset$.

Consequently, $\lim_{n \to \infty} \nu(A_n) = 0$ and $\nu$ is continuous. The uniqueness of the extension $\nu$ is easily verified.

**Maharam's Problem VII:**

If $\nu$ is a submeasure on the Borel sets of $K$ such that for every coordinate set $S \in \mathcal{C}$, $\nu(S) = 1$, must $\nu$ fail to be continuous?

Recall that $K = \prod_{n=1}^{\infty} [K_n]$ where $\lim_{n \to \infty} K_n = \infty$. Let $G_n$ denote the group of permutations on the finite set $[K_n]$ and let $G$ denote the compact topological group

$$G = \prod_{n=1}^{\infty} G_n.$$

We let each $g \in G$ act on $K$ in the obvious way, i.e. if $g \in \prod_{n=1}^{\infty} G_n$ where $g = \langle g_n \rangle$, then

$$g(x) = \langle g_n(x_n) \rangle.$$
Of course, if $S$ is a coordinate set, then so is $g(S)$ whenever $g \in G$. Let $\mu$ denote the Haar probability measure on $G$. Suppose $\nu$ is a continuous submeasure on the Borel sets such that $\nu(S) = 1$ for every $S \in \mathcal{E}$. If we define

$$\nu_0(A) = \int \nu(g(A)) d\mu(g),$$

then $\nu_0$ is a submeasure on the Borel sets such that $\nu_0(S) = 1$ for all $S \in \mathcal{E}$. An application of the Lebesgue dominated convergence theorem easily allows us to verify that $\nu_0$ is continuous. Also (as is easily seen) $\nu_0$ is $G$-invariant, i.e. for every Borel set $A$ and $g \in G$,

$$\nu_0(gA) = \nu_0(A).$$

**Maharam’s Problem VIII:**

If $\nu$ is a $G$-invariant submeasure on the Borel sets of $K$ such that for every coordinate set $S \in \mathcal{E}, \nu(S) = 1$, must $\nu$ fail to be continuous?

**Definition 7.2.**

If $\mathcal{A}$ is a Boolean $\sigma$-algebra and $\nu$ is a continuous submeasure on $\mathcal{A}$ and $a \in \mathcal{A}$ we let $\nu_a$ denote the continuous submeasure on $\mathcal{A}$ defined by

$$\nu_a(b) = \nu(a \land b).$$

**Proposition 7.3.**

If $\nu$ is a strictly positive continuous submeasure on a submeasure algebra $\mathcal{A}$ and $\nu_0$ is a continuous submeasure on $\mathcal{A}$, then for some $a \in \mathcal{A}, \nu_0$ is equivalent to $\nu_a$.

This result is easily proved and implies the following:

**Proposition 7.4.**

If $\nu$ and $\nu_0$ are continuous submeasures on a $\sigma$-algebra $\mathcal{A}$ such that $\nu_0$ is absolutely continuous with respect to $\nu$, then for some $A \in \mathcal{A}, \nu_0$ is equivalent to $\nu_A$.

We now return to the situation where $\nu$ is a continuous $G$-invariant submeasure on the Borel sets of $K$ and $\nu(S) = 1$ for all $S \in \mathcal{E}$. If $E \in cl(K)$, then $E \in \mathcal{A}_m$ for some $m \in N$, where $\mathcal{A}_m$ is the finite algebra generated by the finite collection of coordinate sets \{ $S_{ij}$ : $1 \leq i \leq K_j, 1 \leq j \leq m$ \}. If $S_1, S_2 \in \mathcal{E}_n$ where $m < n$, then

$$\nu(E \cap S_1) = \nu(E \cap S_2)$$

since $\nu$ is $G$-invariant. If for each $n \in N$, we select $S_n \in \mathcal{E}_n$, the sequence

$$< \nu(E \cap S_n) >$$

has a convergent subsequence. Since $cl(K)$ is a countable collection, we may by a Cantor diagonal sequence argument, select an infinite set $L$ of integers such that $\lim_{n \in L} \nu(E \cap S_n)$ exists for all $E \in \mathcal{E}(K)$. We define $\nu_0$ on $\mathcal{E}(K)$ by

$$\nu_0(E) = \lim_{n \in L} \nu(E \cap S_n).$$

Note that $\nu_0 \leq \nu$ on $\mathcal{E}(K)$, so $\nu_0$ is an exhaustive submeasure on $\mathcal{E}(K)$. We also let $\nu_0$ denote its extension to the Borel sets and observe that $\nu_0$ is absolutely continuous with respect to $\nu$ (since $\nu_0 \leq \nu$ on the Borel sets).
Proposition 7.5.
If $S_n \in \mathcal{C}_n$ for each $n \in L$, then

$$\inf_{n \in L} \nu_0(S_n) > 0.$$ 

Proof:
First we observe that $\nu_0$ does not vanish because $\nu_0(K) = \lim_{n \in L} \nu(S_n) = 1$. Thus $\nu_0$ cannot vanish for each $S \in \mathcal{C}_n$. Since $\nu_0$ is $G$-invariant, $\nu_0(S) > 0$ for each $S \in \mathcal{C}_n$. Hence $\nu_0(S) > 0$ for each $S \in \mathcal{C}$. Thus if $\inf_{n \in L} \nu_0(S_n) = 0$ then

$$\lim_{n \in L} \nu_0(S_n) = 0.$$ 

Let us suppose $\lim_{n \in L} \nu_0(S_n) = 0$. By selecting an infinite subset $M$ of $L$ we may suppose that

$$\lim_{n \in M} \nu_0(S_n) = 0.$$ 

Note that $\nu_0(E) = \lim_{n \in M} \nu(E \cap S_n)$ for every $E \in cl(K)$ since $M \subset L$. Since $\lim_{n \in M} \nu_0(S_n) = 0$, $\nu_0$ is not equivalent to $\nu$. Thus $\nu_0$ is equivalent to $\nu_\lambda$ for some Borel set $\lambda$ where $\nu(\lambda) > 0$ and $\nu(\lambda) > 0$. By passing to a further infinite subset of $M$, if necessary, we may suppose that

$$\nu_1(E) = \lim_{n \in M} \nu_\lambda(E \cap S_n)$$

exists for each $E \in cl(K)$. Since $\nu_\lambda \leq \nu$ we have $\nu_1 \leq \nu_0$. Thus $\nu_1(\lambda) = 0$. But also $\nu_1 \leq \nu_\lambda$ so $\nu_1(\lambda) = 0$, i.e. $\nu_1$ vanishes, i.e.

$$\lim_{n \in M} \nu_\lambda(S_n) = 0.$$ 

But $\nu_\lambda$ is equivalent to $\nu_0$, so

$$\lim_{n \in M} \nu_\lambda(S_n) = 0.$$ 

Since $1 = \nu(S_n) \leq \nu_\lambda(S_n) + \nu_\lambda(S_n)$ we have a contradiction.

Notice that since $\inf_{n \in L} \nu_0(S_n) > 0$, the continuous submeasure $\nu_0$ is every bit as pathological as $\nu$, i.e. $\nu_0$ is a $G$-invariant continuous submeasure with no equivalent measure.

Also by replacing our copy of the Cantor set $\prod_{n=1}^{\infty} [K_n]$ with $\prod_{n \in L} [K_n]$ we could assume that $\lim_{n \to \infty} \nu(E \cap S_n)$ exists for every $E \in K$ with $K = \prod_{n \in L} [K_n]$. Henceforth, we assume $\nu$ has this property and

$$\nu_0(E) = \lim_{n \to \infty} \nu(E \cap S_n)$$

for every $E \in cl(K)$. 

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It is easily seen that if $M$ is an infinite subset of $N$ and $S_n \in \mathcal{C}_n$ for each $n \in M$, then the set $\bigcap_{n \in M} \tilde{S}_n$ has cardinality $c$ many disjoint $G$-translates. Hence

$$\nu(\bigcap_{n \in M} S_n) = 0.$$ 

Sets of the form

$$\bigcap_{n \in M} \tilde{S}_n$$

are, of course, much larger.

**Proposition 7.6.**

If $M$ is an infinite subset of $N$ and $S_n \in \mathcal{C}_n$ for each $n \in M$, then

$$\nu_0 \left( \bigcap_{n \in M} \tilde{S}_n \right) = 0.$$ 

**Proof:**

For notational simplicity assume $M = N$. Suppose that

$$\nu_0 \left( \bigcap_{n=1}^{\infty} \tilde{S}_n \right) > \varepsilon > 0.$$ 

Select a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ inductively as follows: once $n_1 < \cdots < n_{k-1}$ have been selected

$$\lim_{n \to \infty} \nu(\tilde{S}_{n_1} \cap \cdots \cap \tilde{S}_{n_k} \cap S_n) = \nu_0(\tilde{S}_{m_1} \cap \cdots \cap \tilde{S}_{m_k}) > \varepsilon.$$ 

Hence there exists $n_k > n_{k-1}$ such that

$$\nu(\tilde{S}_{n_1} \cap \cdots \cap \tilde{S}_{n_{k-1}} \cap S_{n_k}) > \varepsilon.$$ 

This is impossible, since the sequence $\tilde{S}_{n_1} \cap \cdots \cap \tilde{S}_{n_k} \cap S_{n_k}$ is pairwise disjoint.

Any finite positive measure on the Cantor set must give some set of the form $\bigcap_{n \in M} \tilde{S}_n$ positive measure. These sets are nowhere dense. So this is a slight refinement of the fact that every positive finite measure gives some nowhere dense set positive measure.

**Maharam's Problem IX:**

If $\nu_0$ is a continuous submeasure on the Borel sets of $K$, does there exist $M$ infinite and $S_n \in \mathcal{C}_n$ for each $n \in M$ so that

$$\nu_0(\bigcap_{n \in M} \tilde{S}_n) > 0?$$
8. PATHOLOGICAL SUBMEASURES ON cl(K)

Maharam's Problem, in its most tractable form, is really about submeasures on cl(K). The next clue comes from the following scheme for constructing submeasures on cl(K).

Definition 8.1:
Suppose $D$ is a collection of clopen sets which covers $K$ and suppose

$$\phi: D \rightarrow [0, 1].$$

For $E \in \text{cl}(K)$ we let

$$\phi^*(E) = \inf \left\{ \sum_{i=1}^{n} \phi(E_i) : E_1, \ldots, E_n \in D \text{ and } E \subset \bigcup_{i=1}^{n} E_i \right\}$$

Clearly $\phi^*$ is a submeasure on cl(K). Also, if $E \in D$, $\phi^*(E) \leq \phi(E)$. If $\nu$ is a submeasure on cl(K) and we take $D=\text{cl}(K)$, then $\nu^* = \nu$. The apparent need to have $D$ cover $K$ can be avoided by the following convention: if $D$ does not cover $K$, extend $\phi$ to $D \cup \{K\}$ by letting $\phi(K) = 1$. We call the function $\phi$ a weight function and for $E \in D$ we call $\phi(E)$ the weight of $E$. We shall say that $\phi$ is sufficiently heavy if $\phi^*(S) \geq 1$ for every $S \in \mathcal{C}$. We shall say that $\phi$ is sufficiently light if whenever $\varepsilon > 0$ and $< E_n >$ is a pairwise disjoint sequence, there exists $E_n \in D$ such that $\phi(E_n) < \varepsilon$. It is easily seen that if $\phi$ is sufficiently light, then $\phi^*$ is exhaustive. If $\nu$ is a counterexample to Maharam's problem (with $\nu(S) = 1$, each $S \in \mathcal{C}$) then the weight function $\nu$ is both sufficiently heavy and sufficiently light. A weight function $\phi$ which is both sufficiently heavy and sufficiently light produces a counterexample ($\nu = \phi^*$) to Maharam's Problem.

Maharam's Problem X:
Does every weight function fail to be either sufficiently heavy or to be sufficiently light?
For a fixed $\varepsilon > 0$, we shall say that a weight function $\phi$ is lighter than $\varepsilon$ if whenever $< E_n >$ is a pairwise disjoint sequence in cl(K) there exists some $E_N \in D$ such that $\phi(E_n) < \varepsilon$. Of course, if $\phi$ is lighter than $\varepsilon$, then (by an easy argument with subsequences) whenever $< E_n >$ is a pairwise disjoint sequence in cl(K)

$$\lim_{n \to \infty} \phi^*(E_n) \leq \varepsilon$$

We now construct, for each $\varepsilon > 0$, a sufficiently heavy weight function which is lighter than $\varepsilon$. To do this we introduce the notion of complexity of sets. Recall that in our version

$$K = \prod_{n=1}^{\infty} [K_n]$$

of the Cantor set, $\mathcal{C}_n$ denotes the collection of coordinate sets for the n'th coordinate. We let $A_n$ denote the finite algebra generated by $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_n$. We let at($A_n$) denote the atoms of $A_n$. The sets in at($A_n$) are of the form:

$$S_1 \cap S_2 \cap \ldots \cap S_n$$
where each $S_i \in \mathcal{C}_i$, $1 \leq i \leq n$. Also note that $A_1 \subset A_2 \subset \ldots$ and

$$\text{cl}(K) = \bigcup_{n=1}^{\infty} A_n$$

If $E \in \text{cl}(K)$, we say that $E$ has complexity $n$ if $E \in A_n$. If $E \in \text{cl}(K)$ and $n \in \mathbb{N}$ we define $[E]_n$ to be the smallest set in $A_n$ which contains $E$. Alternately,

$$[E]_n = \bigcup \{ A \in \text{at}(A_n) : A \cap E \neq \emptyset \}.$$ 

Of course, $E$ has complexity $n$ if and only if $[E]_n = E$. If $E$ has complexity $n$ and $n < m$, then $E$ also has complexity $m$. We say that the complexity of $E$ is the least $n$ such that $E$ has complexity $n$. We denote this by writing $c(E) = n$. To get a firm grasp of the intuition here, suppose $E$ is a clopen set. Consider the sequence

$$< [E]_i >.$$ 

If the complexity of $E$ is $n$, the sequence is constantly $E$ for $i \geq n$. If $i < n$, we do not know $E$ by knowing $[E]_i$ at the $i$'th stage. $E$ is enigmatic (or complex) at this stage.

Henceforth, we shall say that any finite pairwise disjoint collection $\{E_1, E_2, \ldots, E_n\}$ in $\text{cl}(K)$ is a partition. Technically, $\bigcup_{n=1}^{\infty} \bigcup E_i$ is a finite partition, but the collection $\{E_1, \ldots, E_n\}$ does specify a finite partition. Suppose $\Pi = \{E_1, \ldots, E_n\}$ is a partition and $c(E_i) = m_i$ where $m_1 < m_2 < \ldots < m_n$. We say that $\Pi$ is consistent if whenever $i < j, k$

$$[E_j]_{m_i} = [E_k]_{m_i},$$

i.e. for each $i = 1, 2, \ldots, n-2$, $[E_{i+1}]_{m_i} = [E_{i+2}]_{m_i}$.

We define the notion of a consistent pairwise disjoint sequence similarly.

**Lemma 8.2:**

Each pairwise disjoint sequence in $\text{cl}(K)$ has a consistent subsequence.

**Proof:**

Suppose $< E_n >$ is a pairwise disjoint sequence of nonempty sets in $\text{cl}(K)$. Let $E_{n_1} = E_1$ and let $c(E_{n_1}) = m_1$. Since $A_{m_1}$ is a finite set, there is some $A \in A_{m_1}$ such that

$$\{ n \in \mathbb{N} : [E_n]_{m_1} = A \}$$

is infinite. Let $n_2$ be the first element of this set with $1 = n_1 < n_2$. Continue inductively, in this fashion, to choose $n_1 < n_2 < n_3 < \ldots$.

**Theorem 8.3:**

For each $\varepsilon > 0$ there is a sufficiently heavy weight function that is lighter than $\varepsilon$.  

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Proof:

Let $N$ be an integer such that $1/N < \varepsilon$. Let $D$ denote the collection of sets $E$ in $\text{cl}(K)$ such that there exists a consistent partition $\Pi = \{E_1, E_2, \ldots, E_{3N+1}\}$ with $E = E_{3N+1}$. Define $\phi(E) = 1/N$ if $E \in D$. By the previous lemma $\phi$ is lighter than $\varepsilon$. Let $S \in \mathcal{C}_n$ for some $n$ and let $F_1, \ldots, F_N \in \mathcal{C}$. We shall show that $\{F_1, \ldots, F_N\}$ does not cover $S$, i.e. $\phi(S) \geq 1$. Each $F_i \in \Pi_i$ where

$$\Pi_i = \{E_{i1}, E_{i2}, \ldots, E_{i3N}, F_i\}$$

is consistent. Select $i$ so that $c(E_{i3})$ is the smallest. Without loss of generality we may assume that $c(E_{i3})$ is the smallest. Let $E_{11} = B_1, E_{12} = C_1$ and $E_{13} = D_1$. Now select $i$ among $\{2, 3, \ldots, N\}$ so that $c(E_{is})$ is smallest. We may assume that $c(E_{id})$ is the smallest. We let

$$B_2 = E_{2k}, C_2 = E_{25}, D_2 = E_{26}.$$  

So

$$c(D_1) < c(B_2)$$

and the partitions

$$\{B_1, C_1, D_1, F_1\} \text{ and } \{B_2, C_2, D_2, F_2\} \text{ are consistent.}$$

Continue in this fashion to choose $\{B_i, C_i, D_i\}$ in $\Pi_i$. After renumbering, if necessary, we may suppose that for $1 \leq i \leq N - 1$,

$$c(D_i) < c(B_{i+1}).$$

Also each $\{B_i, C_i, D_i, F_i\}$ is consistent. We now select $A_1 \supset A_2 \supset \ldots A_N = A$ so that

(i) $A_i \in \text{at}(A_k)$ where $k_i = c(D_i)$.

(ii) $A_i \cap F_i = \phi, 1 \leq i \leq N$ (so $A \cap F_i = \phi$).

(iii) $A_i \cap S \neq \phi, 1 \leq i \leq N$ (so $A \cap S \neq \phi$).

Now $S \in \mathcal{C}_m$. So that if $A_o \in \text{at}(A_m)$ then $S \cap A_o \neq \phi$ if $m < n$ and $A_o \cap S \neq \phi$ only if $A_o \subset S$ when $n \leq m$. First let $A_o \in \text{at}(A_1)$ so that $A_o \cap S \neq \phi$. We select $A_1 \in \text{at}(A_{k_1})$ according to the following four cases:

Case 1. $n \leq c(B_1)$

Case 2. $c(B_1) < n \leq c(C_1)$

Case 3. $c(C_1) < n \leq c(D_1)$

Case 4. $c(D_1) < n$.

Case 1

Choose $A'_1 \in \text{at}(A_n)$ so that $A_o \supset A'_1$ and $A'_1 \subset S$. Let $m = c(B_1)$. If $A'_1 \cap C_1 = \phi$, then since $[C_1]_m = [F_1]_m$ and $A'_1 \in A_m, F_1 \cap A'_1 = \phi$. Thus we can choose any $A_1 \in \text{at}(A_{k_1})$ so that $A'_1 \supset A_1$. If $A'_1 \cap C_1 \neq \phi$, choose $A_1 \in A_{k_1}$ such that $A_1 \subset A'_1 \cap C_1$. Since $C_1 \cap F_1 = \phi, A_1 \cap F_1 = \phi$.  

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Case 2

Once again choose $A'_1 \in \sigma(A_n)$ such that $A_o \supset A'_1$ and $A'_1 \subset S$. Let $m = c(C_1)$. If $A'_1 \cap D_1 = \emptyset$, then since $[D_1]_m = [F_1]_m$ and $A'_1 \in A_m$, $A'_1 \cap F_1 = \emptyset$. Any choice of $A_1 \subset A'_1$ with $A_1 \in \sigma(A_k)$ will do. If $A'_1 \cap D_1 \neq \emptyset$, then we may choose $A_1 \subset \sigma(A_k)$ so that $A_1 \subset A'_1 \cap D_1$. Thus $A_1 \cap F_1 = \emptyset$ since $D_1 \cap F_1 = \emptyset$.

Case 3

Let $m = c(C_1)$. By an argument identical to the above we may select $A'_1 \in \sigma(A_m)$ so that $A'_1 \subset A_o$ and $A'_1 \cap F_1 = \emptyset$. Since $m < n$, $A'_1 \cap S \neq \emptyset$. So we can find $A_1 \in \sigma(A_k)$ so that $A_1 \cap S \neq \emptyset$.

Case 4

We can easily find $A_1 \in \sigma(A_k)$ so that $A_1 \subset A_o$ and $A_1 \cap F_1 = \emptyset$. Since $k_1 < n$, $A_1 \cap S \neq \emptyset$.

The choice of $A_2, \ldots, A_N$ proceeds in exactly the same way. In each case $A_{i-1}$ plays the role of $A_o$ above. By (i), (ii) and (iii), $\{F_1, \ldots, F_N\}$ fails to cover $S$.

Notice that if $\nu$ is a submeasure so that whenever $<E_n>$ is pairwise disjoint,

$$\lim_{n \to \infty} \nu(E_n) = 0$$

then $\nu$ is exhaustive. We can produce a pathological submeasure which has

$$\lim_{n \to \infty} \nu(E_n) = 0$$

for pairwise disjoint sequences $<E_n>$ with a uniform bound on $<c(E_n)>$.

Theorem 8.4:

If $<m_i>$ is a strictly increasing sequence of positive integers, there exists a submeasure $\nu$ on $\text{cl}(K)$ such that $\nu(S) \geq 1$ for each $S \in \mathcal{C}$ and if $<E_i>$ is a pairwise disjoint sequence such that $E_i \in A_{m_i}$ for each $i$, then

$$\lim_{n \to \infty} \nu(E_n) = 0$$

The main idea of the proof (which we shall not give) is the following:

Lemma 8.5:

If $<m_i>$ is a strictly increasing sequence of positive integers and $n$ is a positive integer, then there exists $N$ a positive integer such that whenever $\Pi = \{E_1, \ldots, E_N\}$ is a partition with $E_i \in A_{m_i}$, there exists a consistent partition $\Pi'$ with $\Pi' \subset \Pi$ so that $\Pi'$ has at least $n$ elements.
Proof:

Let $X$ denote the set of all $E = < E_i > \in \prod_{i=1}^{\infty} A_{m_i}$ such that $< E_i >$ is a disjoint sequence. Since each $A_{m_i}$ is finite, $\prod_{i=1}^{\infty} A_{m_i}$ is compact in the product topology. It is easily seen that $X$ is a closed subset of the product. Let $X_N$ denote the set of all $E \in X$ such that $< E_1, \ldots, E_N >$ does not have a consistent subsequence of length $n$. Each $X_N$ is closed in $X$ and $X_1 \supset X_2 \supset \ldots$. If $\bigcap_{N=1}^{\infty} X_N \neq \emptyset$, then there is a sequence $< E_i > \in X$ which does not have arbitrarily large consistent partitions. This is impossible, so $X_N = \emptyset$ for some $N$.

The idea of the proof of the theorem is then to use the lemma to construct a weight function similar to the one used for Theorem 8.4. We leave the (nontrivial) details to the reader.

The two examples serve as danger signs to anyone attempting a positive solution to Maharam's Problem. For instance, one cannot prove that there is an absolute constant $\varepsilon > 0$ so that every submeasure $\nu$ (with $\nu(S) \geq 1$ for each $S \in \mathcal{G}$) has a pairwise disjoint sequence $< E_n >$ with $\nu(E_n) \geq \varepsilon$ for each $n$. In other words, some lines of attack do not work, because these examples stand in the way.

It would be very nice if we could press the consistent partition idea further to produce a counterexample to Maharam's Problem. Unfortunately, this is impossible. Notice that such submeasures are $G$-invariant. If $\nu$ is an exhaustive pathological $G$-invariant submeasure we can choose a consistent pairwise disjoint sequence $< E_n >$ with the following properties:

(1) $m_1 < m_2 < \ldots$
(2) $E_m \in A_{m_n}$
(3) $\nu(E_n) < \varepsilon_n$ where $\sum_{n=1}^{\infty} \varepsilon_n = 1/2$.
(4) $[E_{n+1}]_{m_n} = \bigcup_{k=1}^{n} E_k$.

To see how to do this first choose $E_1$ with $\nu(E_1) < \varepsilon_1$ and let $m_1 = c(E_1)$. $E_1$ is a union of $N_1$ many atoms in $A_{m_1}$. Since $\nu$ is a $G$-invariant submeasure, there exists $m_2$ so that

$$\nu(A) < \frac{\varepsilon_2}{m_1}$$

whenever $A \in \operatorname{at}(A_{m_2})$ (note that $A = S_1 \cap \ldots \cap S_{m_2}$). Hence if we select for each atom $A' \in A_{m_1}$ (with $A' \cap E_1 = \emptyset$) an atom $A \subset A'$ with $A \in A_{m_2}$ and let $E_2$ be the union of these atoms, then $\nu(E_2) < \varepsilon_2$. Notice that

$$\bar{E}_1 = [E_2]_{m_1}$$

Continue choosing $E_3, E_4, \ldots$ in this fashion. At last with $< E_n >$ selected satisfying (1)-(4),

$$\left\{ E_1, \ldots, E_n, \bigcup_{i=1}^{n} E_i \right\}$$
is consistent for each \( n \). But

\[
\nu \left( \bigcup_{i=1}^{n} E_i \right) > \frac{1}{2} \text{ for each } n.
\]

The moral is that a "real" submeasure (pathological and exhaustive) does not give small value to a set merely because it is the last term in a long consistent partition.

Let \(< m_k \) be a strictly increasing sequence of positive integers. It can easily be demonstrated that if \( \nu \) is an exhaustive pathological submeasure then there is no uniform rate at which the sequence \( \nu(E_k) \) converges to zero when \( E_k \in \mathcal{A}_{m_k} \) for each \( k \) with \(< E_k \) a pairwise disjoint sequence. However, there is a uniform rate at which small terms of the sequence \( \nu(E_k) \) must appear.

**Proposition 8.6:**

Suppose \( \nu \) is an exhaustive submeasure on \( cl(K) \). If \(< m_k \) is a strictly increasing sequence of positive integers and \( \varepsilon > 0 \), there exists an integer \( N \) and \( \beta_1 \geq \beta_2 \geq \ldots \geq \beta_N \) so that for any partition \( \{E_1, \ldots, E_N\} \) with each \( E_k \in \mathcal{A}_{m_k} \)

\[
\sum_{k=1}^{N} \beta_k \nu(E_k) \leq \varepsilon
\]

where

\[
\sum_{k=1}^{N} \beta_k = 1.
\]

**Proof:**

Let \( X \) denote the set of \( E =< E_k > \) in \( \prod_{k=1}^{\infty} \mathcal{A}_{m_k} \) so that \( < E_k > \) is pairwise disjoint. \( X \) is a closed subset of the compact product space, so \( X \) is compact. Let \( P(X) \) denote the regular Borel probability measures on \( X \).

For \( \mu \in P(X) \), let

\[
g_n(\mu) = \int \frac{1}{n} \sum_{k=1}^{\infty} \nu(E_k) \mu(E).
\]

Since \( \nu \) is exhaustive,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \nu(E_k) = 0
\]

for each \( E \in X \). By the bounded convergence theorem,

\[
\lim_{n \to \infty} g_n(\mu) = 0
\]

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for each $\mu \in P(X)$. Each $g_n$ is weak* continuous on $P(X)$. Thus when we let

$$f_n = \min\{g_1, g_2, \ldots, g_n\}$$

each $f_n$ is continuous on $P(X)$ and the sequence $< f_n >$ decreases to zero. $P(X)$ is weak* compact, so by Dini's theorem the sequence $< f_n >$ converges to zero uniformly. Hence, there exists an integer $N$ so that $f_N(\mu) \leq \varepsilon$ for each $\mu \in P(X)$.

For $E = \{E_1, \ldots, E_N\}$ a partition in $\prod_{k=1}^{N} A_{m_k}$ define

$$\phi_E(n) = \frac{1}{n} \sum_{k=1}^{n} \nu(E_k).$$

Each $\phi_E$ is a function on $\Omega = [N] = \{1, 2, \ldots, N\}$. Let $\mathcal{C}$ be the convex hull of the functions $\phi_E$. If $\phi \in \mathcal{C}$, then since $f_N \leq \varepsilon$, there exists $n \in \Omega$ so that

$$\phi(n) \leq \varepsilon.$$

Applying Proposition 3.2 to the functions $\{\varepsilon - \phi: \phi \in \mathcal{C}\}$ on $\Omega$ we obtain a probability measure $\alpha$ on $\Omega$ so that

$$\int \varepsilon - \phi(n) d\alpha(n) \geq 0$$

for each $\phi \in \mathcal{C}$, i.e.

$$\int \phi(n) d\alpha(n) = \sum_{n=1}^{N} \phi(n) \alpha(n) \leq \varepsilon.$$

For $E = \{E_1, \ldots, E_N\}$ a partition with each $E_k \in A_{m_k}$, letting $\phi = \phi_E$, we have

$$\sum_{n=1}^{N} \alpha(n) \cdot \left( \frac{1}{n} \sum_{k=1}^{n} \nu(E_k) \right) \leq \varepsilon.$$

Now let $\beta_k = \sum_{n=k}^{N} \frac{\alpha(n)}{n}$. It is not too difficult to apply the above proposition to obtain:

**Proposition 8.7:**

Suppose $\nu$ is an exhaustive submeasure on $\text{cl}(K)$. If $< m_k >$ is a strictly increasing sequence of positive, then there exists a nonnegative decreasing sequence $< \beta_k >$ with

$$\sum_{k=1}^{\infty} \beta_k = \infty$$

so that if $< E_k >$ is a pairwise disjoint sequence with each $E_k \in A_{m_k}$, then

$$\sum_{k=1}^{\infty} \beta_k \nu(E_k) \leq 1.$$
9. A COMBINATORIAL APPROACH

We now consider a nonconstructive approach to the question of the existence (or nonexistence) of pathological exhaustive submeasures. In this section we let $\mathcal{A}$ denote $\text{cl}(K)$. The key idea is that the set of weight functions on $\mathcal{A}$ is just $[0,1]^\mathcal{A}$. Since $\mathcal{A}$ is countable, this is a copy of the Hilbert Cube. In particular, $[0,1]^\mathcal{A}$ is a compact convex set. Note that the set of extreme points of $[0,1]^\mathcal{A}$ is the set $\{0,1\}^\mathcal{A}$. By Choquet's Theorem, for every $\phi \in [0,1]^\mathcal{A}$, there is a Borel probability measure $\mu$ on $\{0,1\}^\mathcal{A}$ so that for each $E \in \mathcal{A}$,

$$\phi(E) = \int x(E) d\mu(x).$$

The use of Choquet's Theorem here is a little heavy-handed, because it is easy to show that there exists a sequence $<x_n>$ with each $x_n \in \{0,1\}^\mathcal{A}$ so that

$$\phi(E) = \sum_{n=1}^{\infty} 2^{-n} x_n(E)$$

for every $E \in \mathcal{A}$. So one could take $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$.

This observation is useful because it converts the problem of finding a set function satisfying certain inequalities to the problem of finding a probability measure satisfying certain inequalities. Of course, Proposition 3.2 now comes into play. If $E \in \mathcal{A}$, we let $e(E)$ denote the coordinate function on $\{0,1\}^\mathcal{A}$ defined by

$$e(E)(x) = \begin{cases} 1 & \text{if } x(E) = 1 \\ 0 & \text{if } x(E) = 0. \end{cases}$$

Of course, each $e(E)$ is continuous on $\{0,1\}^\mathcal{A}$. In order to have $\phi$ be sufficiently heavy, we will need

$$(9.1) \quad \int \sum_{i=1}^{p} e(E_i) d\mu \geq 1$$

whenever $\{E_1, \ldots, E_p\}$ is a cover of some $S \in \mathcal{C}$. If $M = <m_k>$ is an increasing sequence of integers and $<E_k>$ is a disjoint collection with each $E_k \in \mathcal{A}_{m_k}$, then we want

$$\sum_{k=1}^{\infty} \beta_k \phi(E_k) \leq 1$$

or equivalently

$$\sum_{k=1}^{p} \beta_k \phi(E_k) \leq 1$$

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for each $p$. Here $<\beta_k>$ is a decreasing sequence (dependent upon $M$) so that

$$\sum_{k=1}^{\infty} \beta_k = \infty.$$  

Technically, $\beta = \beta(M)$ and $\beta_k = \beta(m,k)$. We shall refer to such functions of the two variables $M$ and $k$ as just $\beta$. For a specific partition $\Pi = \{E_1, \ldots, E_p\}$ with $E_k \in A_m$, we shall write $\beta(E_k)$ for $\beta_k$ or $\beta(m,k)$. So for $\phi$ to be sufficiently light we shall require that

$$\int \sum_{k=1}^{p} \beta(E_k) e(E_k) d\mu \leq 1$$

for each partition $\Pi = \{E_1, \ldots, E_p\}$.

A collection $U = \{E_1, \ldots, E_r\}$ in $\text{cl}(K)$ will be called a cover if for some $S \in \mathcal{C}$,

$$S \subseteq \bigcup_{i=1}^{r} E_i.$$  

The existence of a weight function which is both sufficiently heavy and sufficiently light is equivalent by inequalities (9.1) and (9.2) to the existence of a $\beta$ and a probability measure $\mu$ on $\{0,1\}^A$ so that functions of the form:

$$\sum_{E \in U} e(E) - 1$$

and $1 - \sum_{E \in \mathcal{E}} \beta(E) e(E)$ have nonnegative integral with respect to $\mu$. By Proposition 3.2 we have the following:

**Theorem 9.1:**

There is an exhaustive pathological submeasure if and only if for some choice of $\beta$ and every sequence $(U_1, \ldots, U_m)$ of covers and every sequence $(\Pi_1, \ldots, \Pi_n)$ of partitions, there exists $x \in \{0,1\}^A$ so that

$$n - m + \sum_{i=1}^{m} \sum_{E \in U_i} e(E)(x) - \sum_{k=1}^{n} \sum_{F \in \Pi_k} \beta(F) e(F)(x) \geq 0.$$  

Rephrasing the above condition, the nonexistence of an exhaustive pathological submeasure implies that for every choice of $\beta$ there is a finite sequence $(U_1, \ldots, U_m)$ of covers and $(\Pi_1, \ldots, \Pi_n)$ of partitions so that the function

$$n - m + \sum_{i=1}^{m} \sum_{E \in U_i} e(E) - \sum_{k=1}^{n} \sum_{F \in \Pi_k} \beta(F) e(F) < 0.$$
Let us make a few observations about this. First of all, suppose a clopen set \( E \) appears in the sequence \((U_1, \ldots, U_m)\) more frequently than in the sequence \((\Pi_1, \ldots, \Pi_n)\). Then any \( x \in \{0,1\}^A \) which maximizes the above function will have \( x(E) = 1 \). By removing a \( U_i \) (with \( E \in U_i \)) the resulting function is still maximized by an \( x \in \{0,1\}^A \) with \( x(E) = 1 \) and is still negative, i.e.,

\[
\sum_{E \in U_i} c(E)(x) \geq 1 \text{ and } m \text{ is replaced by } m - 1.
\]

By a sequence of steps in which covers \( U_i \) are removed, we may assume that each \( E \in \text{cl}(K) \) appears among the partitions \((\Pi_1, \ldots, \Pi_n)\) at least as frequently as among the covers \((U_1, \ldots, U_m)\). Also we can assume that \( m > n \). If \( m \leq n \) then letting \( x(E) = 0 \) for every \( E \in A \) would provide a nonnegative value for the function. In Theorem 9.1 one need only consider sums satisfying these two conditions.

Theorem 9.1 is a sword with two edges. The theorem could be used to prove the existence or to prove the nonexistence of exhaustive pathological submeasures. Let us consider the latter possibility. Suppose that for each \( \beta \) we can produce sequences \((U_1, \ldots, U_m)\) and \((\Pi_1, \ldots, \Pi_n)\) to make a negative-valued function. Also suppose each set occurs among the partitions as frequently as among the covers. We shall show that each \( x \in K \) gives rise, in a natural way, to a one-to-one function.

Let \( x \in K \). Then \( \{x\} = \bigcap_{i=1}^{\infty} S_i \) where \( < S_i > \) is a sequence of coordinate sets with each \( S_i \subseteq U_i \). Let

\[
D(x) = \{j \in [m] : U_j \text{ is a cover of some } S_i\}.
\]

Consider all maps \( \psi : D(x) \to [n] \) that satisfy the following:

**Rule:** if \( \psi(j) = k \), then there exists a set \( E \in U_j \cap \Pi_k \) such that \( x \in E \).

It is easy to find a function \( \psi \) satisfying the rule. If \( U_j \) covers some \( S_i \), then since \( x \in S_i \), \( x \in E \) for some \( E \in U_j \). Choose a partition \( \Pi_k \) so that \( E \in \Pi_k \) and let \( \psi(j) = k \). Now for each \( j \in D(x) \), select \( E_j \in U_j \) so that \( x \in E_j \). The sets \( < E_j ; j \in D(x) > \) may not all be distinct. Suppose \( E_{j_1} \neq E_{j_2} \). If \( \psi(j_1) = \psi(j_2) = k \) then \( E_{j_1}, E_{j_2} \in \Pi_k \). But this is impossible since \( x \in E_{j_1} \cap E_{j_2} \) and \( \Pi_k \) is a partition. In other words, if \( E_{j_1} \neq E_{j_2} \) then \( \psi(j_1) \neq \psi(j_2) \). It is now easy to see that one can select a \( \psi \) satisfying the rule so that \( \psi \) is one-to-one.

The set \( D(x) \) can be partitioned into sets of the form:

\[
\{j \in D(x) : E_j = E\}.
\]

Since each \( E \) occurs at least as frequently in the partition sequence as in the cover sequence, we can define a one-to-one map from \( \{j \in D(x) : E_j = E\} \) to the set \( \{k \in [n] : E \in \Pi_k\} \). The resulting \( \psi \) is one-to-one.
Using the above observation as motivation, we show how one can construct sequences \((U_1, \ldots, U_m)\) of covers and \((\Pi, \ldots, \Pi_p)\) of partitions so that each \(E\) arises more frequently among the partitions (when this is possible). Let \(X = \prod_{n=1}^{t} [K_n]\) and let \(Y\) be a finite set. Suppose that for every \(n, 1 \leq n \leq t, P_n\) is a collection of disjoint sets in \(Y\) and that \(P_n\) is divided into collections \(P_{1n}, \ldots, P_{kn}\). For every \(x \in X\) let

\[
D(x) = \bigcup_{n=1}^{t} P_{xn}.
\]

Suppose that for every \(x \in X\) there exists \(\psi_x: D(x) \rightarrow P\) such that

1. if \(M \in P_{xn}\) then \(\psi_x(M) \in M\).
2. \(\psi_x\) is one-to-one
3. if \(x, y \in X\) such that \(x(1) = y(1), \ldots, x(s) = y(s)\) then \(\psi_x\) agrees with \(\psi_y\) on
   \[
P_{xz(t)}, \ldots, P_{xz(s)}.
   \]

If \(M \in P_{in}\) and \(M = \{p_1, p_2, \ldots, p_r\}\) we let \(c(M) = \{E_1, \ldots, E_r\}\) where \(E_i = \{x: x(n) = i\}\) and \(\psi_x(M) = p_i\). Each \(c(M)\) is a cover (of \(S_{in}\)). If \(p \in P\), we let \(\Pi(p)\) denote the collection of sets \(E\) where \(E\) is of the form: \(\{x \in X: \psi_x(M) = p\}\) for some \(M\) (in some \(P_{in}\)). It is easily verified (using the fact that each \(\psi_x\) is one-to-one) that each \(\Pi(p)\) is a partition. Condition (3) guarantees that when \(E = \{x: x(n) = i\}\) and \(\psi_x(M) = p\) with \(M \in P_{in}\), then \(E \in A_n\). This helps to control the complexity of sets in the partitions.

Typically, the set \(Y\) is a collection of subsets of \([N]\) and each \(P_n = \{F \in Y: n \in F\}\). In this case, if \(F = \{m_1, m_2, \ldots, m_s\}\), \(\Pi(F) = \{E_1, E_2, \ldots, E_s\}\) with \(E_i \in A_{m_i}\).

Using the above ideas, it is possible to devise a purely combinatorial formulation of Maharam's Problem. The difficulty comes largely from the consistency condition (3) (or variations of this). The study of the selection of one-to-one functions is called matching theory. It is ironic that Maharam's Problem, which begins as a purely measure theoretic problem, can be reformulated as a purely combinatorial problem, a complex problem in matching theory.
REFERENCES


10. The Scottish Book.

Some problems concerning integral operators.
by Pawel Szeptycki *) (Lawrence, KS)

We will discuss some unsolved problems concerning integral operators. Some of these problems were stated in print previously, some may be new.

An integral operator is formally defined by the formula \( \int_S k(t,s)x(s) ds = Kx(t) \), where \( S \) and \( T \) are \( \sigma \)-finite measure spaces and \( K \) is considered as an operator form \( L^0(S) \) to \( L^0(T) \) - the spaces of complex valued measurable functions on \( S \) and \( T \), furnished with the metric topologies of convergence in measure on all subsets of finite measure.

The proper domain of \( K \) is defined as the subspace of \( L^0(S) \) on which the integral makes sense, i.e. \( D_K = \{ u; |K||u|(t) = |S|k(t,s)||u(s)| ds < \infty \text{ a.e.} \} \). \( D_K \) furnished with the graph topology of the sublinear operator \( u \rightarrow |K||u| \) is an F-space. Except for some very special cases it is not an F-space with the graph topology of the operator \( K \).

The extended domain of \( K \) appears in the process of extending the operator \( K \) by continuity to subspaces of \( L^0(S) \) other than \( D_K \). Let \( V \) be a topological vector subspace of \( L^0(S) \). We say that \( K \) is \( V \)-semiregular if \( |K||D_K \cap V| < \text{dense in } V \), and \( K: D_K \cap V \rightarrow L^0(T) \) is continuous from \( V \). If this is the case, \( K \) can be extended to a linear operator \( K_V \) from \( V \) to \( L^0(T) \), which of course need not be an integral operator.

An example of this situation is provided by the Fourier transform; here \( S=\mathbb{T}=\mathbb{R}, k(t,s) = e^{ist} \), \( D_K = L^1 \) and \( K \) is \( L^2 \)-s.r.

A particularly clear situation arises when we consider solid spaces. It turns out that in this case among all the (solid) spaces satisfying the conditions i),ii) of semiregularity, there is one, referred to as the extended domain of \( K \), \( D_K \) which is maximal in the sense that it contains continuously all the solid subspaces of \( L^0 \) with respect to which \( K \) is semi-regular.

The extension of \( K \) to \( D_K \) is denoted by \( K \) and all extensions \( K_V \) of \( K \) to solid subspaces \( V \) are obtained by restricting to \( V \) the maximal extension \( K \).

The extended domain turns out to be an F-space; this may seem somewhat surprising since the maximality property persists with respect to all solid topological spaces (with respect to which \( K \) is s.r.) and not only metric spaces. A more detailed discussion of the extended domain and some usable descriptions thereof can be found in [1] and [2].

Besides of their intrinsic interest the proper domains and the extended domains provide examples of interesting function and sequence spaces.

One of the problems that arise is that of determination of the spaces \( D_K \) and \( D_K \) for specific kernels and classes of kernels. In the case when the kernel \( k \) varies moderately, e.g. if it is of constant sign, the extended domain is the same as the proper domain and no extensions (to solid spaces) are possible.

The equality \( D_K = D_K \) holds also when the space \( T \) is purely atomic.

In the case when \( K \) is the Fourier transform, the extended domain is the amalgam space \( L^2(L^1) = \{ u \in L^1_{loc}(\mathbb{R}); \Sigma (\int_{l_n} |u| ds)^2 < \infty \}, \) where \( l_n = (n,n+1) \), \( n = 0,1,-1,2,-2,\ldots \). By a

*) The contents of this paper was presented by Iwo Labuda.
suitable change of variables this allows one to find the extended domains of operators with kernels of the form \( k(t,s) = e^{its} \) where \( m \) is any positive integer: these are compressed amalgams obtained by replacing in the above definition the intervals \( I_n \) by the intervals \( \sqrt{n}, \sqrt{n+1} \). This result can be extended (with a different proof, of course), see [3], to kernels of the form \( e^{ip(t,s)} \) where \( p \) is an arbitrary real polynomial. This is in agreement with the intuition that the size of the extended domain is in a sense determined by the rate of oscillations of the kernel about 0 - faster the oscillations, larger the extended domain.

It would be of some interest to see how far one could carry the above idea, in particular, one would like to determine extended domains for oscillating kernels which are not smooth; for example, is the extended domain corresponding to the kernel \( \text{sign(sin(ts))} \) the same as that for \( \sin(ts) \), i.e. \( \chi^2(L^1) \)?

Also, what (if anything) corresponds to the above description of the extended domain for the kernel \( e^{ip(t,s)} \) in dimensions larger than 1 (i.e. \( t \in \mathbb{R}^n, s \in \mathbb{R}^m \))? - Note that the characterization of the extended domain of the Fourier transform as \( \chi^2(L^1) \) is valid in the case of an arbitrary locally compact abelian group.

If \( k \) is a kernel of absolute value 1, for instance the kernel of the Fourier transform, and if \( k_1 \) is positive kernel, what is then the extended domain of the operator with the kernel \( k_1k \)? In the case when \( k_1 \) is the Fourier kernel one would expect the answer to be \( \chi^2(D_{K_1}) \) (an appropriate definition of this space is of course part of the problem).

So much for the determination of the extended domains.

Except for the case of purely atomic measure the space \( L^0 \) is not locally convex and the constructions of the proper domain and of the extended domains appear to lead to spaces which a priori are not locally convex. The lack of local convexity can not be avoided in general: in the case when \( k=0 \) we have \( D_K = L^0 \). However there are indications that if the nullspace \( N_K \) of the operator \( u \to |K| |u| \) is \( \{0\} \) then the spaces \( D_K \) and \( D_K' \) may be locally convex.

Iwo Labuda proved that if \( N_K = \{0\} \) then \( D_K' \) is locally convex if and only if there is a locally convex vector space \( V \) continuously included in \( L^0(T) \) such that \( K: D_K' \to V \). This result would be enhanced by an example of a kernel \( k \) for which there is no such subspace \( V \) or, what may amount to the same, by a characterization of all kernels \( k \) for which there exist such \( V \)-s.

If \( T \) is purely atomic the condition of Labuda is clearly satisfied by \( V = L^0(T) \); it is also easy to write down the family of seminorms defining the topology of \( D_K \), without reference to the Labuda’s condition. As already mentioned above, in this case \( D_K \) and \( D_K' \) coincide.

In this context one may want to look at the following example. Let \( S \) be the set of all positive integers and \( T = (0,1) \). Represent every \( s \) in \( S \) in the form \( s = 2^m + k \) where \( 0 \leq k < 2^m \), and let \( k(t,s) \) be the characteristic function of the interval \( (k2^{-m}, (k+1)2^{-m}) \) (this sequence probably has a name). The domain \( D_K \) is a sequence space which contains sequences which may be of arbitrary growth, provided they contain sufficiently many zeros. It would be of interest to find a direct description of this space and of its topology - by the general construction of the proper domain this is given by the F-norm \( (a_g) \to \rho_T(\sum |a_g| k(t,s)) \), where \( \rho_T \) is the F-norm.
defining their topology of $L^0(T)$. Is this topology locally convex? An equivalent question is the following. Is the space \{ $\Sigma a_k(\tau)\Delta_2$; $\Sigma|a_k|\Delta_2$ a.e.$\} \text{ locally convex in the topology of convergence in measure on } (0,1)?($Added after this paper was presented: a straightforward argument shows that this topology is not locally convex).

The next question has been asked on several occasions and deals with compactness of integral operators. It is known that if $p\geq q\geq 1$ and if $K$ is an integral operator such that $|K|$ is a (necessarily bounded) operator from $L^p$ to $L^q$ then $K$ is compact operator from $L^p$ to $L^q$. Can the condition that $|K|$ acts from $L^p$ to $L^q$ be replaced by the conditions that $L^p$ is contained in $D_K$ and that $KL^p$ is contained in $L^q$? The question can probably be answered by somebody knowledgable about factorisation of operators between spaces in question.

As the last problem we would like to mention the question of compatibility raised implicitly in [1].

Let $L_1, L_2$ be topological vector spaces let $D_K$ be a subspace of $L_1$ and let $K: D_A \to L_2$ be a linear operator, not necessarily an integral operator. As before, we say that $A$ is semiregular relative to a subspace $V$ of $L_1$ if the conditions i, ii appearing above are satisfied. If this is the case, then we denote by $K_V$ the continuous extension of $K$ to $V$. If $\mathcal{M}$ is a family of subspaces of $L_1$, then we say that $K$ satisfies the compatibility condition with respect to $\mathcal{M}$ for every $V \in \mathcal{M}$ such that $K$ is both $V_1$ and $V_2$-semiregular, and for every $u \in V_1 \cap V_2$, we have $K_{V_1}u = K_{V_2}u$.

The existence of the extended domain for an integral operator shows that such an operator satisfies the compatibility condition with respect to the class of solid subspaces of $L^0$.

The problem is to determine a compatibility classes for integral operators, which would be larger than the class of solid subspaces of $L^0$.

It seems likely that the compatibility property may persist for a larger class of subspaces of $L^0$, e.g. subspaces satisfying some majorization properties (like $u \in V \Rightarrow |u| \in V$).

It is possible that such classes compatibility may exist for some restricted classes of kernels - e.g. $K$ is universally compatible (i.e. compatible relative to all subspaces of $L^0$) if and only if $K$ is closable (see [1]). Also there is no a priori reason for existence of a maximal element in a class of compatibility for a fixed kernel.

References: