ANALYSIS SEMINAR

Remainder Maps

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Abstract: *We study* remainder maps $R_f(z) = \sum_{k=1}^{\infty} f(z+k)$, *associated to convergent series defined by complex rational maps* $f \in \mathbb{F}(X)$, *with coefficients in an arbitrary subfield* $\mathbb{F} \subset \mathbb{C}$. *All such maps are meromorphic on* $\mathbb{C}$, *form a vector space over* $\mathbb{F}$, *and generate what we call the* remainder field $\mathbb{F}(\mathcal{R}_{\mathbb{F}})$ of ground $\mathbb{F}$. *For remainder maps, we prove a criterion of algebraic independence over* $\mathbb{F}(X)$, *stated in terms of* $\mathbb{F}$*-linear combinations. It turns out that any* R_f *is either a member of* $\mathbb{F}(X)$, *or is transcendental, and so the field extension* $\mathbb{F}(\mathcal{R}_{\mathbb{F}})/\mathbb{F}(X)$ *is purely transcendental. We show that any nonzero* φ *from the remainder field has a* degree $\mathrm{d}(\varphi) \in \mathbb{Z}$, *for which* $\frac{\varphi(x)}{x^{\mathrm{d}(\varphi)}}$ *has a nonzero limit (as* $x \to \infty$*) in* $\mathbb{F}$. *This yields a natural asymptotic expansion of* φ. *Consequently, the remainder field can be extended to the field* $\mathbb{F}((X^{-1}))$ *of reverse formal Laurent series, and rational approximants of any order (as good as we want) exist. For remainder maps, finding rational approximants reduces to the same problem for the* fundamental case *of* $f = \frac{1}{X^2}$. *We thus find the explicit formal series of any remainder map. In the general case, the difference* $\frac{1}{R_f(x)} - W_f(x)$ *tends to* 0 *for some unique* $W_f \in \mathbb{F}[X]$, *called here the* inverse polynomial *of* R_f. *In the real case, the convergence is monotonic and leads to sharp estimates. Iterating the construction leads to higher order inverse polynomials, with exponentially increasing degrees. In our example, the error made by replacing the sum of the series* $\sum_{n \geq 1} \frac{1}{n^2}$ *(the fundamental case) by a corrected* n*th partial sum, is less than* 10^{-4} *for* $n = 1$, *than* 10^{-15} *for* $n = 5$, *and than* 10^{-32} *for* $n = 20$. *Our theory also includes the* alternating remainder maps $\widehat{R}_f(z) := \sum_{k=1}^{\infty} (-1)^{k-1} f(2z+k)$. *Students are welcome.*