ON STRUCTURE OF UPPER SEMICONTINUITY

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Let Y be a topological space, $\mathcal{A}$, $\mathcal{B}$ families of its subsets. We write $\mathcal{B}\#\mathcal{A}$ and say that $\mathcal{B}$ and $\mathcal{A}$ *mesh*, if $A \cap B \neq \emptyset$ for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$. $\mathcal{B}$ is *compactoid relative to* A, if each filter meshing with $\mathcal{B}$ has a cluster point in A.

Let X be another topological space and let $F : X \rightrightarrows Y$ be a set-valued map. F is said to be *upper semicontinuous at* $x \in X$ (*usc at* x), if, for each open set V containing $F(x)$, there exists a neighborhood U of x such that $F(U) \subset V$. F is *upper semicontinuous* (usc) if it is upper semicontinuous at x for each $x \in X$.

We write $\mathcal{B} \rightsquigarrow A$ and say that $\mathcal{B}$ *aims at* A, if, for each neighborhood V of A there exists B in $\mathcal{B}$ such that $B \subset V$. Let $\mathcal{U} = \mathcal{U}(x)$ be the filter of neighborhoods of x. The family $\{F(U) : U \in \mathcal{U}\}$ is obviously a base of a filter on Y. F is usc at x if and only if $F(\mathcal{U}) \rightsquigarrow F(x)$.

A set A contained in Y is called a *cap* (of upper semicontinuity) *of* F *at* x_0 if the map defined by setting $F(x_0) = A$ and keeping other values of F intact, is usc at x_0.

The *external part or map* (of F at x_0) is the map $E(\,.\,) := F(\,.\,) \setminus F(x_0)$. Hence $E(\mathcal{U})$ denotes the image filter base of $\mathcal{U} = \mathcal{U}(x_0)$ by the external map, that is, $\{F(U) \setminus F(x_0) : U \in \mathcal{U}(x_0)\}$. We call it *external filter base* (of F at x_0).

Let $\mathcal{U}(x)$ be the filter of neighborhoods of $x \in X$. *Active boundary* of F at x_0 is the adherence of $E(\mathcal{U})$, that is,

$$\operatorname{Frac} F(x_0) = adhE(\mathcal{U}) = \bigcap_{U \in \mathcal{U}(x_0)} \overline{\{F(U) \setminus F(x)\}},$$

The name $\operatorname{Frac} F(x_0)$ originates from French '**fr**ontière **ac**tive'. The notion was introduced by Dolecki in order to prove that, if X, Y are metric spaces and F is usc at x_0, then its active boundary is a *compact cap* (for F at x_0). The theorems of this type are sometimes called *Choquet-Dolecki theorems*. They are equivalent with the fact that the corresponding *external filter base is compactoid.*

The compactoidness of $E(\mathcal{U})$ seems to be the ultimate strengthening of upper semicontinuity. We will show that it takes place under considerably weaker assumptions about spaces X and Y than previously thought.

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