ON STRUCTURE OF UPPER SEMICONTINUITY

IWO LABUDA

Let $Y$ be a topological space, $A$, $B$ families of its subsets. We write $B \# A$ and say that $B$ and $A$ mesh, if $A \cap B \neq \emptyset$ for each $A \in A$ and each $B \in B$. $B$ is compactoid relative to $A$, if each filter meshing with $B$ has a cluster point in $A$.

Let $X$ be another topological space and let $F : X \rightrightarrows Y$ be a set-valued map.

$F$ is said to be upper semicontinuous at $x \in X$ (usc at $x$), if, for each open set $V$ containing $F(x)$, there exists a neighborhood $U$ of $x$ such that $F(U) \subset V$. $F$ is upper semicontinuous (usc) if it is upper semicontinuous at $x$ for each $x \in X$.

We write $B \Rightarrow A$ and say that $B$ aims at $A$, if, for each neighborhood $V$ of $A$ there exists $B$ in $B$ such that $B \subset V$. Let $U = U(x)$ be the filter of neighborhoods of $x$. The family $\{F(U) : U \in U\}$ is obviously a base of a filter on $Y$. $F$ is usc at $x$ if and only if $F(U) \Rightarrow F(x)$.

A set $A$ contained in $Y$ is called a cap (of upper semicontinuity) of $F$ at $x_0$ if the map defined by setting $F(x_0) = A$ and keeping other values of $F$ intact, is usc at $x_0$.

The external part or map (of $F$ at $x_0$) is the map $E(\cdot) := F(\cdot) \setminus F(x_0)$. Hence $E(U)$ denotes the image filter base of $U = U(x_0)$ by the external map, that is, $\{F(U) \setminus F(x_0) : U \in U(x_0)\}$. We call it external filter base (of $F$ at $x_0$).

Let $U(x)$ be the filter of neighborhoods of $x \in X$. Active boundary of $F$ at $x_0$ is the adherence of $E(U)$, that is,

$$\text{Frac } F(x_0) = \text{adh } E(U) = \bigcap_{U \in U(x_0)} \{F(U) \setminus F(x)\},$$

The name $\text{Frac } F(x_0)$ originates from French ‘frontière active’. The notion was introduced by Dolecki in order to prove that, if $X, Y$ are metric spaces and $F$ is usc at $x_0$, then its active boundary is a compact cap (for $F$ at $x_0$). The theorems of this type are sometimes called Choquet-Dolecki theorems. They are equivalent with the fact that the corresponding external filter base is compactoid.

The compactoidness of $E(U)$ seems to be the ultimate strengthening of upper semicontinuity. We will show that it takes place under considerably weaker assumptions about spaces $X$ and $Y$ than previously thought.